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Title: An adjoint Green's function approach to model instabilities in a duct with mean flow

Abstract: This document gives a detailed derivation of the integral governing equation for the velocity potential in the following setup: A finite duct with mean flow and frequency-dependent end conditions (described by reflection coefficients) houses a compact unsteady heat source; thermoacoustic instabilities arise in the duct because of the interaction between the sound field in the duct and the unsteady heat release rate from the heat source. The derivation of the governing integral equation is based on the acoustic analogy equation, i.e. the convected wave equation with a source term that represents the heat release rate. A series of mathematical steps are performed. These include an adjoint approach with a test function. This test function is shown to be the adjoint form of the tailored Green's function of the flow duct. It is calculated with a generalised function approach and turns out to be a superposition of duct modes. Expressions for the frequencies and amplitudes of these modes are found. they depend on the reflection coefficients, duct length and speed of the flow. The final result of this document is the integral governing equation for the velocity potential; this contains the adjoint Green's function, the heat release rate and initial conditions. It can be solved with a straightforward time-stepping approach.

Keywords: flow duct, unsteady heat source, acoustic analogy equation, adjoint Green's function, integral governing equation

An adjoint Green's function approach to model instabilities in a duct with mean flow

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Notation

c : speed of sound

g : direct Green's function

G : adjoint Green's function

H: Heaviside function

k_+, k_- : wave numbers

n : mode number

$q(t)$: measure for the fluctuating part of the global heat release rate (per unit volume)

R_0, R_L : pressure reflection coefficients

t : observer time

t' : source time (τ in Jiasen's notation)

$T_?$: terminal time

$\bar{u}$: speed of the mean flow

x : observer point

x' : source point (ξ in Jiasen's notation)

γ : specific heat ratio

ϕ : fluctuating part of the velocity potential

φ_0 : initial value of the velocity potential

φ'_0 : initial value of the acoustic pressure

$\bar{\rho}$: mean density

ω : angular frequency

time-dependence for the direct problem: $e^{-i\omega t}$

Starting point

We consider a duct with ends at $x = 0$ and $x = L$, described by pressure reflection coefficients R_0 and R_L, respectively (see figure 1). The speed of the flow is $\bar{u}$, and the speed of sound is c. The Mach number $M = \bar{u}/c$ is assumed to be smaller than 1. The setup is treated as 1-D.

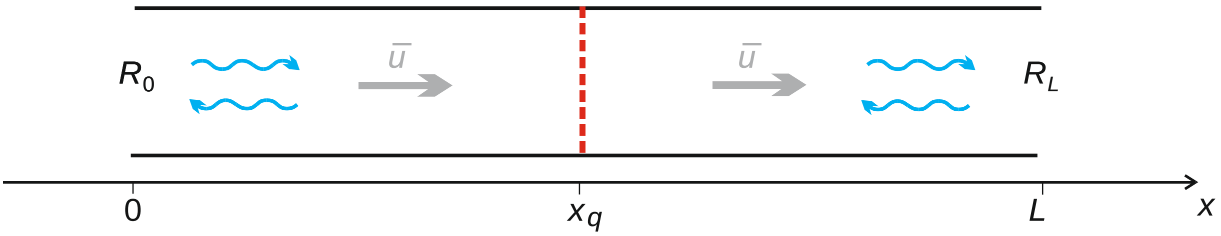

Figure 1: Schematic illustration of a flow duct with compact unsteady heat source at x_q

PDE:

$$\frac{\partial^2 \phi}{\partial t^2} + 2\bar{u}\frac{\partial^2 \phi}{\partial t \partial x} - \left(c^2 - \bar{u}^2\right)\frac{\partial^2 \phi}{\partial x^2} = -\frac{\gamma - 1}{\bar{\rho}} q(t)\delta(x - x_q) \tag{1}$$

initial conditions:

$$\phi(x,t)\big|_{t=0} = \varphi_0\, \delta(x - x_q) \tag{2a}$$

$$\left[\frac{\partial \phi}{\partial t} + \bar{u}\frac{\partial \phi}{\partial x}\right]_{t=0} = \varphi'_0\, \delta(x - x_q) \tag{2b}$$

The boundary conditions are given in the frequency-domain by $R_0(\omega)$, $R_L(\omega)$.

Our time-dependence is $e^{-i\omega t}$.

Step 1: Determine the characteristic equation and the wave numbers

Results (see Appendix for Step 1):

characteristic equation: $$\left(c^2-\bar{u}^2\right)k^2+2\bar{u}\omega k-\omega^2=0 \qquad (3)$$

wave numbers: $$k_+=k_1=\frac{\omega}{c+\bar{u}}, \quad k_-=-k_2=\frac{\omega}{c-\bar{u}} \qquad (4a,b)$$

Step 2: Acoustic field in the frequency-domain

Results (see Appendix for Step 2)

$$\hat{\phi}(x,\omega)=\begin{cases} A_-(R_0\,\mathrm{e}^{\mathrm{i}k_+x}+\mathrm{e}^{-\mathrm{i}k_-x}) & \text{upstream of the heat source} \\ B_+(\mathrm{e}^{\mathrm{i}k_+(x-L)}+R_L\mathrm{e}^{-\mathrm{i}k_-(x-L)}) & \text{downstream of the heat source} \end{cases} \qquad (5)$$

Step 3: Try out the adjoint approach

Result (see Appendix for Step 3)

$$\int_{t'=0}^{T_?}\int_{x'=0}^{L}\left[\frac{\partial^2 G}{\partial t'^2}+2\bar{u}\frac{\partial^2 G}{\partial t'\partial x'}-\left(c^2-\bar{u}^2\right)\frac{\partial^2 G}{\partial x'^2}\right]\phi(x',t')\mathrm{d}x'\mathrm{d}t'+$$

$$+\int_{x'=0}^{L}\underbrace{\left[G\left(\frac{\partial\phi}{\partial t'}+\bar{u}\frac{\partial\phi}{\partial x'}\right)-\phi\left(\frac{\partial G}{\partial t'}+\bar{u}\frac{\partial G}{\partial x'}\right)\right]_{t'=0}^{T_?}}_{=\mathrm{BT1}}\mathrm{d}x'+$$

$$\underbrace{+\int_{t'=0}^{T_?}\left[\bar{u}\left(G\frac{\partial\phi}{\partial t'}-\phi\frac{\partial G}{\partial t'}\right)-(c^2-\bar{u}^2)\left(G\frac{\partial\phi}{\partial x'}-\phi\frac{\partial G}{\partial x'}\right)\right]_{x'=0}^{L}\mathrm{d}t'}_{=\mathrm{BT2}}=$$

$$=-\frac{\gamma-1}{\bar{\rho}}\int_{t'=0}^{T_?}G(x_q,x,t',t)q(t')\mathrm{d}t' \qquad (6)$$

At this stage, $G(x',x,t',t)$ is a test function, and $T_?$ is a terminal time. Both are yet to be determined.

Step 4: Define the test function *G* (as far as possible)

We define the test function $G(x',x,t',t)$ in such a way that equation (6) becomes as simple as possible and gives an integral equation for the acoustic field $\phi(x,t)$.

Results (see Appendix for Step 4):

PDE:

$$\frac{\partial^2 G}{\partial t'^2} + 2\bar{u}\frac{\partial^2 G}{\partial t' \partial x'} - \left(c^2 - \bar{u}^2\right)\frac{\partial^2 G}{\partial x'^2} = \delta(x'-x)\delta(t'-t) \tag{7}$$

terminal conditions:

$$G(x',x,t',t) = 0 \quad \text{at } t' = T_? \tag{8a}$$

$$\frac{\partial G}{\partial t'} + \bar{u}\frac{\partial G}{\partial x'} = 0 \quad \text{at } t' = T_? \tag{8b}$$

We call $G(x',x,t',t)$ the "*adjoint Green's function*", following the notation of [Morse and Feshbach 1953, section 7.5] (rather than that of [Greenberg 1978, section 22.5], which is out of line with all the literature on Green's functions known to me).

The remaining terms of (6) are:

$$\phi(x,t) = -\frac{\gamma-1}{\bar{\rho}}\int_{t'=0}^{T_?} G(x_q,x,t',t)\,q(t')\,\mathrm{d}t' + \left[\varphi'_0\,G(x',x,t',t) - \varphi_0\left(\frac{\partial G}{\partial t'} + \bar{u}\frac{\partial G}{\partial x'}\right)\right]_{\substack{x'=x_q\\ t'=0}} + \mathrm{BT2}. \tag{9}$$

$G(x',x,t',t)$ is not fully defined at this stage, because the boundary conditions have not been specified yet.

Step 5: Consider the direct Green's function *g*(*x*,*x'*,*t*,*t'*) and calculate it

$g(x,x',t,t')$ is defined as the impulse response of the flow duct and has been calculated in the Appendix for Step 5.The impulse is fired at time t' from a hypothetical point source at x'. The response is measured by an observer at location x and time t. The measured response does not depend on t or t' individually, but on the time lapsed since the impulse, $t-t'$.

PDE in the time-domain:

$$\frac{\partial^2 g}{\partial t^2}+2\bar{u}\frac{\partial^2 g}{\partial t\partial x}-\left(c^2-\bar{u}^2\right)\frac{\partial^2 g}{\partial x^2}=\delta(x-x')\delta(t-t') \tag{10}$$

causality conditions:

$$g(x,x',t-t')=0 \quad \text{for } t<t' \tag{11a}$$

$$\frac{\partial g}{\partial t}+\bar{u}\frac{\partial g}{\partial x}=0 \quad \text{for } t<t' \tag{11b}$$

PDE in the frequency domain:

$$\omega^2\hat{g}(x,x',\omega)+2\bar{u}\,\mathrm{i}\omega\frac{\partial\hat{g}}{\partial x}+\left(c^2-\bar{u}^2\right)\frac{\partial^2\hat{g}}{\partial x^2}=-\delta(x-x') \tag{12}$$

boundary conditions (given in the frequency domain):

near $x=0$: $$\hat{g}(x,x',\omega)=A_-(x',\omega)[R_0\,\mathrm{e}^{\frac{\mathrm{i}\omega}{c+\bar{u}}x}+\mathrm{e}^{-\frac{\mathrm{i}\omega}{c-\bar{u}}x}] \tag{13a}$$

near $x=L$: $$\hat{g}(x,x',\omega)=B_+(x',\omega)[\mathrm{e}^{\frac{\mathrm{i}\omega}{c+\bar{u}}(x-L)}+R_L\mathrm{e}^{-\frac{\mathrm{i}\omega}{c-\bar{u}}(x-L)}] \tag{13b}$$

g and $\hat{g}$ are a Fourier transform pair:

$$g(x,x',t-t')=\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\hat{g}(x,x',\omega)\mathrm{e}^{-\mathrm{i}\omega(t-t')}\mathrm{d}\omega \tag{14a}$$

$$\hat{g}(x,x',\omega)=\int_{t=-\infty}^{\infty}g(x,x',t-t')\mathrm{e}^{\mathrm{i}\omega(t-t')}\mathrm{d}t \tag{14b}$$

Result for $g(x,x',t,t')$ (see Appendix for Step 5)

$$g(x,x',t-t')=\mathrm{H}(t-t')\sum_{n=1}^{\infty}\mathrm{Re}\left[\frac{g_n(x,x',\omega_n)}{\omega_n F'(\omega_n)}\mathrm{e}^{-\mathrm{i}\omega_n(t-t')}\right] \tag{15}$$

where

$$g_n(x,x',\omega_n) = \begin{cases} \psi(x',\omega_n) b(x',\omega_n) a(x,\omega_n) & \text{for} \quad x < x' \\ \psi(x',\omega_n) a(x',\omega_n) b(x,\omega_n) & \text{for} \quad x > x' \end{cases} \tag{16}$$

and

$$F(\omega) = -1 + R_0 R_L \mathrm{e}^{\mathrm{i}\omega \frac{2cL}{c^2 - \bar{u}^2}}, \tag{17a}$$

$$a(x,\omega) = R_0\, \mathrm{e}^{\mathrm{i}k_+ x} + \mathrm{e}^{-\mathrm{i}k_- x}, \tag{17b}$$

$$b(x,\omega) = \mathrm{e}^{\mathrm{i}k_+ (x-L)} + R_L\, \mathrm{e}^{-\mathrm{i}k_- (x-L)}, \tag{17c}$$

$$\psi(x,\omega) = -\frac{1}{c} \mathrm{e}^{-\mathrm{i}(k_+ - k_-)x} \mathrm{e}^{-\mathrm{i}k_+ L}. \tag{18}$$

The Heaviside $\mathrm{H}(t-t')$ function in (15) expresses the *causality* of the Green's function:

$$\mathrm{H}(t-t') = \begin{cases} 0 & \text{for} \quad t < t', \text{ i.e. before the impulse} \\ 1 & \text{for} \quad t > t', \text{ i.e. after the impulse} \end{cases} \tag{19}$$

Step 6: Determine the adjoint of the frequency-domain Green's function $\hat{g}(x,x',\omega)$

Results (see Appendix for Step 6):

PDE:

$$\omega^2\hat{G}(x,x',\omega)-2\bar{u}\,\mathrm{i}\omega\frac{\partial\hat{G}}{\partial x}+\left(c^2-\bar{u}^2\right)\frac{\partial^2\hat{G}}{\partial x^2}=-\delta(x-x') \tag{20}$$

boundary conditions:

$$\hat{G}(x,x',\omega)=\begin{cases}\tilde{A}_-(x',\omega)\,[R_0\,\mathrm{e}^{\frac{\mathrm{i}\omega}{c-\bar{u}}x}+\mathrm{e}^{-\frac{\mathrm{i}\omega}{c+\bar{u}}x}] & \text{near} \quad x=0\\ \tilde{B}_+(x',\omega)\,[\mathrm{e}^{\frac{\mathrm{i}\omega}{c-\bar{u}}(x-L)}+R_L\mathrm{e}^{-\frac{\mathrm{i}\omega}{c+\bar{u}}(x-L)}] & \text{near} \quad x=L\end{cases} \tag{21}$$

Step 7: Find the relationship between $G(x',x,t',t)$ **and** $\hat{G}(x',x,\omega)$

Result (see Appendix for Step 7):

$$G(x',x,t',t)=\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\hat{G}(x',x,\omega)\,\mathrm{e}^{-\mathrm{i}\omega(t-t')}\,\mathrm{d}\omega\,, \tag{22}$$

i.e. they are a Fourier transform pair.

Step 8: Show that the boundary term BT2 in Step 3 is zero

The calculations are given in Appendix for Step 8 and confirm that

$$\mathrm{BT2}=0\,. \tag{23}$$

Step 9: Find the relationship between $G(x,x',t,t')$ **and** $g(x,x',t-t')$

Result (see Appendix for Step 9):

$$G(x',x,t',t)=g(x,x',t,t')\,. \tag{24}$$

This can be interpreted as a generalised reciprocity theorem.

Step 10: Give the full result for the adjoint Green's function $G(x,x',t,t')$

Since we have calculated $g(x,x',t-t')$ in Step 5, we obtain $G(x,x',t,t')$ simply by swapping over the variables in (15). This gives

$$\mathrm{G}(x',x,t',t) = \mathrm{H}(t'-t)\sum_{n=0}^{\infty} \mathrm{Re}\left[\frac{g_n(x',x,\omega_n)}{\omega_n F'(\omega_n)} \mathrm{e}^{-\mathrm{i}\omega_n(t'-t)}\right] \tag{25}$$

where

$$g_n(x',x,\omega_n) = \begin{cases} \psi(x,\omega_n) b(x,\omega_n) a(x',\omega_n) & \text{for } x' < x \\ \psi(x,\omega_n) a(x,\omega_n) b(x',\omega_n) & \text{for } x' > x \end{cases} \tag{26}$$

and

$$F(\omega) = -1 + R_0 R_L \mathrm{e}^{\mathrm{i}\omega \frac{2cL}{c^2-\bar{u}^2}}, \tag{27a}$$

$$a(x,\omega) = R_0\,\mathrm{e}^{\mathrm{i}k_+ x} + \mathrm{e}^{-\mathrm{i}k_- x}, \tag{27b}$$

$$b(x,\omega) = \mathrm{e}^{\mathrm{i}k_+(x-L)} + R_L\,\mathrm{e}^{-\mathrm{i}k_-(x-L)}, \tag{27c}$$

$$\psi(x,\omega) = \frac{1}{c}\mathrm{e}^{-\mathrm{i}(k_+-k_-)x}\mathrm{e}^{-\mathrm{i}k_+ L}. \tag{28}$$

The Heaviside function $\mathrm{H}(t'-t)$ in (25) expresses the *anti-causality* (also called *causality in reverse time*) of the adjoint Green's function:

$$\mathrm{H}(t'-t) = \begin{cases} 1 & \text{for} \quad t < t',\ \text{i.e. before the impulse} \\ 0 & \text{for} \quad t > t',\ \text{i.e. after the impulse} \end{cases} \tag{29}$$

Step 11: Fix the terminal time $T_?$ and give the final version of the integral equation

Results (see Appendix for Step 11):

The only meaningful choice is $T_? = t$.

integral equation:

$$\phi(x,t) = -\frac{\gamma-1}{\bar{\rho}}\int_{t'=0}^{t} G(x_q,x,t',t)q(t')\,\mathrm{d}t' + \left[\varphi'_0\, G(x',x,t',t) - \varphi_0\left(\frac{\partial G}{\partial t'} + \bar{u}\frac{\partial G}{\partial x'}\right)\right]_{\substack{x'=x_q \\ t'=0}} \tag{30}$$

Appendix for Step 1: Determine the characteristic equation and the wave numbers

We consider the duct shown in Figure 1 without the heat source, i.e. we solve the homogeneous version of Eq. (1) for the acoustic velocity potential $\phi(x,t)$,

$$\frac{\partial^2 \phi}{\partial t^2} + 2\bar{u}\frac{\partial^2 \phi}{\partial t \partial x} - \left(c^2 - \bar{u}^2\right)\frac{\partial^2 \phi}{\partial x^2} = 0 . \tag{1.1}$$

The frequency-domain equivalent of (1.1) for the velocity potential $\hat{\phi}(x,\omega)$ is (assuming a time-dependence $\mathrm{e}^{-\mathrm{i}\omega t}$)

$$\omega^2 \hat{\phi} + 2\bar{u}(\mathrm{i}\omega)\frac{\partial \hat{\phi}}{\partial x} + \left(c^2 - \bar{u}^2\right)\frac{\partial^2 \hat{\phi}}{\partial x^2} = 0 . \tag{1.2}$$

We solve this with the trial solution

$$\hat{\phi} = \phi_0 \mathrm{e}^{\mathrm{i}kx}, \qquad \frac{\partial \hat{\phi}}{\partial x} = \mathrm{i}k\, \phi_0 \mathrm{e}^{\mathrm{i}kx} = \mathrm{i}k\,\hat{\phi}, \qquad \frac{\partial^2 \hat{\phi}}{\partial x^2} = -k^2\, \phi_0 \mathrm{e}^{\mathrm{i}kx} = -k^2 \hat{\phi} . \tag{1.3a,b,c}$$

Substitution into (1.2) gives

$$\left(c^2 - \bar{u}^2\right)k^2 + 2\bar{u}\omega k - \omega^2 = 0 . \tag{1.4}$$

This is a quadratic equation for k, which has solutions

$$k_{1,2} = \frac{-2\bar{u}\omega \pm \sqrt{4\bar{u}^2\omega^2 + 4\,\omega^2\left(c^2 - \bar{u}^2\right)}}{2\left(c^2 - \bar{u}^2\right)} = \omega\frac{-\bar{u} \pm c}{c^2 - \bar{u}^2} = \omega\frac{\pm c - \bar{u}}{(c+\bar{u})(c-\bar{u})} \tag{1.5}$$

or

$$k_1 = \omega\frac{1}{c+\bar{u}}, \quad k_2 = \omega\frac{(-1)}{c-\bar{u}} \quad . \tag{1.6a,b}$$

We introduce the notation

$$k_+ = k_1 = \frac{\omega}{c+\bar{u}} \text{ (wave number of wave travelling with the flow),} \tag{1.7a}$$

$$k_- = -k_2 = \frac{\omega}{c-\bar{u}} \text{ (wave number of wave travelling against the flow).} \tag{1.7b}$$

Appendix for Step 2: Acoustic field in the frequency-domain

The acoustic velocity potential in the flow-duct can be written as a superposition of a forward and backward travelling waves, with wave numbers given by (1.7a,b),

$$\hat{\phi}(x,\omega) = \begin{cases} A_+ e^{ik_+ x} + A_- e^{-ik_- x} & \text{upstream of the heat source} \\ B_+ e^{ik_+ (x-L)} + B_- e^{-ik_- (x-L)} & \text{downstream of the heat source} \end{cases} \tag{2.1}$$

where A_+, A_-, B_+ and B_- are (generally complex) amplitudes of the velocity potential.

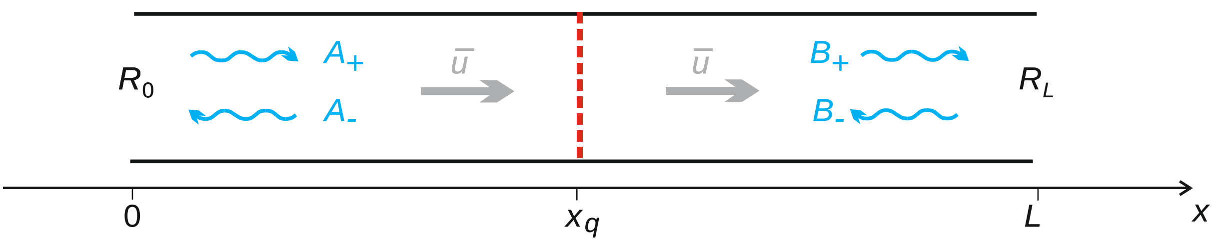

Figure 2.1: Schematic illustration of a flow duct with mean velocity $\bar{u}$, acoustic waves with amplitudes A_+, A_-, B_+, B_-, and an unsteady heat source at x_q

The waves are reflected at the tube ends with reflection coefficients R_0 and R_L.

At $x = 0$, we have

$$R_0 = \left. \frac{A_+ e^{ik_+ x}}{A_- e^{-ik_- x}} \right|_{x=0} \quad \Rightarrow \quad A_+ = A_- R_0 . \tag{2.2}$$

At $x = L$, we have

$$R_L = \left. \frac{B_- e^{-ik_- (x-L)}}{B_+ e^{ik_+ (x-L)}} \right|_{x=L} \quad \Rightarrow \quad B_- = B_+ R_L \qquad . \tag{2.3}$$

We can then write for (2.1)

$$\hat{\phi}(x) = \begin{cases} A_- (R_0 e^{ik_+ x} + e^{-ik_- x}) & \text{upstream of the heat source} \\ B_+ (e^{ik_+ (x-L)} + R_L e^{-ik_- (x-L)}) & \text{downstream of the heat source} \end{cases} \tag{2.4}$$

Appendix for Step 3: Try out the adjoint approach

We consider the governing PDE (1) for the acoustic field $\phi(x,t)$, write it in terms of the variables x',t', multiply it by a test function $G(x',x,t',t)$ (yet to be defined), integrate $\int_{t'=0}^{T_?}\int_{x'=0}^{L} \dots G(x',x,t',t)\dots \mathrm{d}x'\mathrm{d}t'$, and rewrite with integration by parts.

This leads to (see Appendix A in [Wei et al 2023])

$$\int_{t'=0}^{T_?}\int_{x'=0}^{L}\left[\frac{\partial^2 G}{\partial t'^2}+2\bar{u}\frac{\partial^2 G}{\partial t'\partial x'}-\left(c^2-\bar{u}^2\right)\frac{\partial^2 G}{\partial x'^2}\right]\phi(x',t')\mathrm{d}x'\mathrm{d}t'+$$

$$+\int_{x'=0}^{L}\underbrace{\left[G\left(\frac{\partial\phi}{\partial t'}+\bar{u}\frac{\partial\phi}{\partial x'}\right)-\phi\left(\frac{\partial G}{\partial t'}+\bar{u}\frac{\partial G}{\partial x'}\right)\right]_{t'=0}^{T_?}}_{=\mathrm{BT1}}\mathrm{d}x'+$$

$$\underbrace{+\int_{t'=0}^{T_?}\left[\bar{u}\left(G\frac{\partial\phi}{\partial t'}-\phi\frac{\partial G}{\partial t'}\right)-\left(c^2-\bar{u}^2\right)\left(G\frac{\partial\phi}{\partial x'}-\phi\frac{\partial G}{\partial x'}\right)\right]_{x'=0}^{L}\mathrm{d}t'}_{=\mathrm{BT2}}=$$

$$=-\frac{\gamma-1}{\bar{\rho}}\int_{t'=0}^{T_?}G(x_q,x,t',t)q(t')\mathrm{d}t' \qquad (3.1)$$

At this stage, $T_?$ is an unspecified "terminal time", which will be determined in Step 11..

Appendix for Step 4: Define the test function *G* (as far as possible)

We define the test function $G(x',x,t',t)$ in such a way that equation (6) becomes as simple as possible and gives an integral equation for the acoustic field $\phi(x,t)$.

If $G(x,x',t,t')$ satisfies the PDE:

$$\frac{\partial^2 G}{\partial t'^2}+2\bar{u}\frac{\partial^2 G}{\partial t'\partial x'}-\left(c^2-\bar{u}^2\right)\frac{\partial^2 G}{\partial x'^2}=\delta(x'-x)\delta(t'-t)\,, \tag{4.1}$$

then the first integral in (6) reduces to $\phi(x,t)$. If we further impose the terminal conditions

$$G(x',x,t',t)=0 \quad \text{at } t'=T_? \ , \tag{4.2a}$$

$$\frac{\partial G}{\partial t'}+\bar{u}\frac{\partial G}{\partial x'}=0 \quad \text{at } t'=T_? \ , \tag{4.2b}$$

then the terms at $t'=T_?$ in BT1 in (6) vanish,

$$\mathrm{BT1}=\left[\underbrace{G}_{=0}\left(\frac{\partial\phi}{\partial t'}+\bar{u}\frac{\partial\phi}{\partial x'}\right)-\phi\underbrace{\left(\frac{\partial G}{\partial t'}+\bar{u}\frac{\partial G}{\partial x'}\right)}_{=0}\right]_{t'=T_?}-\left[G\left(\frac{\partial\phi}{\partial t'}+\bar{u}\frac{\partial\phi}{\partial x'}\right)-\phi\left(\frac{\partial G}{\partial t'}+\bar{u}\frac{\partial G}{\partial x'}\right)\right]_{t'=0}=0 \tag{4.3}$$

We can rewrite the remaining terms in BT1 with the initial conditions (2a,b) for ϕ to get

$$\mathrm{BT1}=-\left[\varphi'_0\,\delta(x'-x_q)G(x',x,t',t)-\varphi_0\,\delta(x'-x_q)\left(\frac{\partial G}{\partial t'}+\bar{u}\frac{\partial G}{\partial x'}\right)\right]_{t'=0} \tag{4.4}$$

The integral of BT1 in (6) then becomes

$$\int_{x'=0}^{L}\mathrm{BT1}\,\mathrm{d}x'=$$

$$=-\int_{x'=0}^{L}\left[\varphi'_0\,\delta(x'-x_q)G(x',x,t',t)-\varphi_0\,\delta(x'-x_q)\left(\frac{\partial G}{\partial t'}+\bar{u}\frac{\partial G}{\partial x'}\right)\right]_{t'=0}\mathrm{d}x'=$$

$$=-\left[\varphi'_0\,G(x',x,t',t)-\varphi_0\left(\frac{\partial G}{\partial t'}+\bar{u}\frac{\partial G}{\partial x'}\right)\right]_{\substack{x'=x_q\\ t'=0}} \tag{4.5}$$

Eq. (6) reduces with (4.1) and (4.5) to

$$\phi(x,t)-\left[\varphi'_0\,G(x',x,t',t)-\varphi_0\left(\frac{\partial G}{\partial t'}+\bar{u}\frac{\partial G}{\partial x'}\right)\right]_{\substack{x'=x_q\\ t'=0}}+\mathrm{BT2}=-\frac{\gamma-1}{\bar{\rho}}\int_{t'=0}^{T_?}G(x_q,x,t',t)q(t')\,\mathrm{d}t' \tag{4.6}$$

Appendix for Step 5: Consider the direct Green's function *g* and calculate it

Appendix 5.1: Calculation of the eigenfrequencies in the flow duct

We consider the flow duct without any source, so instead of (5), we can write

$$\hat{\phi}(x,\omega) = A_+ \mathrm{e}^{\mathrm{i}k_+ x} + A_- \mathrm{e}^{-\mathrm{i}k_- x} \tag{5.1}$$

for the velocity potential anywhere in the duct.

The waves are reflected at the tube ends with reflection coefficients R_0 and R_L.

At $x = 0$, we have

$$R_0 = \left.\frac{A_+ \mathrm{e}^{\mathrm{i}k_+ x}}{A_- \mathrm{e}^{-\mathrm{i}k_- x}}\right|_{x=0} \quad \Rightarrow \quad A_+ = A_- R_0 . \tag{5.2}$$

At $x = L$, we have

$$R_L = \left.\frac{A_- \mathrm{e}^{-\mathrm{i}k_- x}}{A_+ \mathrm{e}^{\mathrm{i}k_+ x}}\right|_{x=L} \quad \Rightarrow \quad A_- = A_+ R_L \mathrm{e}^{\mathrm{i}k_+ L + \mathrm{i}k_- L} \; . \tag{5.3}$$

Eqs. (5.2) and (5.3) can be written as a homogeneous matrix equation for the amplitudes A_+ and A_-,

$$\begin{bmatrix} 1 & -R_0 \\ R_L \mathrm{e}^{\mathrm{i}(k_+ + k_-)L} & -1 \end{bmatrix} \begin{bmatrix} A_+ \\ A_- \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{5.4}$$

The determinant of the 2×2 matrix in (5.4) must be zero, and this leads to the characteristic equation

$$0 = \begin{vmatrix} 1 & -R_0 \\ R_L \mathrm{e}^{\mathrm{i}(k_+ + k_-)L} & -1 \end{vmatrix} = -1 + R_0 R_L \mathrm{e}^{\mathrm{i}(k_+ + k_-)L} . \tag{5.5}$$

The wave numbers $k_+ + k_-$ can be expressed in terms of the frequency with (1.7a, b),

$$k_+ + k_- = \frac{\omega}{c + \bar{u}} + \frac{\omega}{c - \bar{u}} = \frac{\omega(c - \bar{u}) + \omega(c + \bar{u})}{(c + \bar{u})(c - \bar{u})} = \frac{2\omega c}{c^2 - \bar{u}^2} . \tag{5.6}$$

(5.5) can now be written as

$$F(\omega) = 0 , \tag{5.7a}$$

with

$$F(\omega) = -1 + R_0 R_L \mathrm{e}^{\mathrm{i}\omega \frac{2cL}{c^2 - \bar{u}^2}} . \tag{5.7b}$$

The solution of (5.7) gives the eigenfrequencies $\omega_1, \omega_2, \ldots$

Appendix 5.2: Calculation of the frequency-domain Green's function

We consider the same flow-duct as in Figure 1, but instead of a heat source at x_q, there is a hypothetical point source at x' (see figure 5.1).

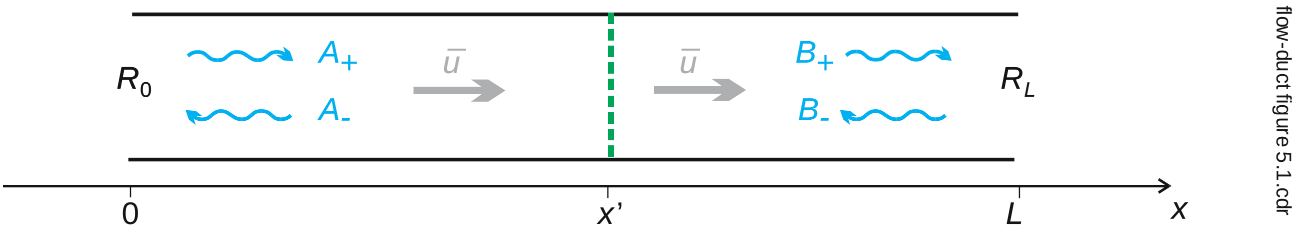

Figure 5.1: Flow-duct with hypothetical point source at x'.

The time-domain Green's function, denoted by $g(x,x',t,t')$, is the response (measured by an observer at point x and time t) to a point source at x' firing an impulsive signal at time t'. Given this physical meaning, we can conclude that g does not depend on t or t' individually, but only on the difference $t-t'$, i.e. the time lapsed since the impulse. We can therefore write its functional dependence as $g(x,x',t-t')$. Its governing equation is

$$\frac{\partial^2 g}{\partial t^2}+2\bar{u}\frac{\partial^2 g}{\partial t\partial x}-\left(c^2-\bar{u}^2\right)\frac{\partial^2 g}{\partial x^2}=\delta(x-x')\delta(t-t'). \tag{5.8}$$

We introduce the Fourier transform of $g(x,x',t-t')$, denoted by $\hat{g}(x,x',\omega)$,and given by

$$g(x,x',t-t')=\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\hat{g}(x,x',\omega)\mathrm{e}^{-\mathrm{i}\omega(t-t')}\mathrm{d}\omega. \tag{5.9}$$

Similarly, we write for the delta function [Dowling and Ffowcs Williams 1983]

$$\delta(t-t')=\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\hat{\delta}(\omega)\mathrm{e}^{-\mathrm{i}\omega(t-t')}\mathrm{d}\omega, \tag{5.10a}$$

and

$$\hat{\delta}(\omega)=1. \tag{5.10b}$$

By applying the Fourier transform to (5.8), we obtain the governing equation for $\hat{g}(x,x',\omega)$,

$$\omega^2\hat{g}(x,x',\omega)+2\bar{u}(\mathrm{i}\omega)\frac{\partial\hat{g}}{\partial x}+\left(c^2-\bar{u}^2\right)\frac{\partial^2\hat{g}}{\partial x^2}=-\delta(x-x'). \tag{5.11}$$

We use as trial solution for (5.11)

$$\hat{g}(x,x',\omega)=\begin{cases}A_+(x',\omega)\mathrm{e}^{\mathrm{i}k_+x}+A_-(x',\omega)\mathrm{e}^{-\mathrm{i}k_-x} & \text{for } x<x' \\ B_+(x',\omega)\mathrm{e}^{\mathrm{i}k_+(x-L)}+B_-(x',\omega)\mathrm{e}^{-\mathrm{i}k_-(x-L)} & \text{for } x>x'\end{cases} \tag{5.12}$$

A_+, A_-, B_+ and B_- are four "velocity potential amplitudes" (they depend on the source position x') that are to be determined. By applying the boundary conditions at the tube ends,

$$x = 0: \qquad A_+ = R_0 A_- \tag{5.13a}$$

$$x = L: \qquad B_- = R_L B_+ \tag{5.13b}$$

we can eliminate two of the amplitudes and rewrite (5.12) as

$$\hat{g}(x,x',\omega) = \begin{cases} A_-(x',\omega)\overbrace{\left[R_0\,e^{ik_+x} + e^{-ik_-x}\right]}^{a(x,\omega)} & \text{for } x < x' \\ B_+(x',\omega)\underbrace{\left[e^{ik_+(x-L)} + R_L\,e^{-ik_-(x-L)}\right]}_{b(x,\omega)} & \text{for } x > x' \end{cases} \tag{5.14}$$

We introduce the abbreviations $a(x,\omega)$ and $b(x,\omega)$ as indicated in (5.14) above.

A_- and B_+ will now be determined in such a way that the governing equation (5.11) is satisfied. To this end, we write (5.14) in terms of the Heaviside function,

$$\hat{g}(x,x',\omega) = \mathrm{H}(x'-x)A_-(x',\omega)\left[R_0\,e^{ik_+x} + e^{-ik_-x}\right] + \mathrm{H}(x-x')B_+(x',\omega)\left[e^{ik_+(x-L)} + R_L\,e^{-ik_-(x-L)}\right] \tag{5.15}$$

and differentiate, using

$$\frac{\partial \mathrm{H}(x-x')}{\partial x} = \delta(x-x') \quad \text{and} \quad \frac{\partial \mathrm{H}(x'-x)}{\partial x} = -\delta(x-x'). \tag{5.16a,b}$$

This gives

$$\begin{aligned} \frac{\partial \hat{g}}{\partial x} = {} & -\delta(x-x')A_-\left[R_0\,e^{ik_+x'} + e^{-ik_-x'}\right] + \mathrm{H}(x'-x)A_-\left[ik_+R_0\,e^{ik_+x} - ik_-\,e^{-ik_-x}\right] + \\ & +\delta(x-x')B_+\left[e^{ik_+(x'-L)} + R_L\,e^{-ik_-(x'-L)}\right] + \mathrm{H}(x-x')B_+\left[ik_+\,e^{ik_+(x-L)} - R_L\,ik_-\,e^{-ik_-(x-L)}\right] \end{aligned} \tag{5.17a}$$

$$\begin{aligned} \frac{\partial^2 \hat{g}}{\partial x^2} = {} & -\delta'(x-x')A_-\left[R_0\,e^{ik_+x'} + e^{-ik_-x'}\right] - \delta(x-x')A_-\left[ik_+R_0\,e^{ik_+x'} - ik_-\,e^{-ik_-x'}\right] + \\ & + \mathrm{H}(x'-x)A_-\left[-k_+^2R_0\,e^{ik_+x} - k_-^2\,e^{-ik_-x}\right] + \\ & +\delta'(x-x')B_+\left[e^{ik_+(x'-L)} + R_L\,e^{-ik_-(x'-L)}\right] + \delta(x-x')B_+\left[ik_+\,e^{ik_+(x'-L)} - ik_-R_L\,e^{-ik_-(x'-L)}\right] + \\ & + \mathrm{H}(x-x')B_+\left[-k_+^2\,e^{ik_+(x-L)} - k_-^2R_L\,e^{-ik_-(x-L)}\right] \end{aligned} \tag{5.17b}$$

This is substituted into (5.11),

$$
\begin{aligned}
&H(x'-x)\Big\{\omega^2 A_-\Big[\cancel{R_0\,e^{ik_+x}}+\cancel{e^{-ik_-x}}\Big]+2\bar{u}\,i\omega A_-\Big[\cancel{ik_+R_0\,e^{ik_+x}}-\cancel{ik_-\,e^{-ik_-x}}\Big]+\\
&\qquad\qquad+(c^2-\bar{u}^2)A_-\Big[\cancel{-k_+^2R_0\,e^{ik_+x}}-\cancel{k_-^2\,e^{-ik_-x}}\Big]\Big\}+\\
&H(x-x')\Big\{\ \omega^2B_+\Big[\cancel{e^{ik_+(x-L)}}+\cancel{R_L\,e^{-ik_-(x-L)}}\Big]+2\bar{u}\,i\omega B_+\Big[\cancel{ik_+\,e^{ik_+(x-L)}}-\cancel{ik_-R_L\,e^{-ik_-(x-L)}}\Big]+\\
&\qquad\qquad+(c^2-\bar{u}^2)B_+\Big[\cancel{-k_+^2\,e^{ik_+(x-L)}}-\cancel{k_-^2R_L\,e^{-ik_-(x-L)}}\Big]\Big\}+\\
&\delta(x-x')\Big\{2\bar{u}\,i\omega(-A_-)\Big[R_0\,e^{ik_+x'}+e^{-ik_-x'}\Big]+2\bar{u}\,i\omega B_+\Big[e^{ik_+(x'-L)}+R_L\,e^{-ik_-(x'-L)}\Big]+\\
&\qquad(c^2-\bar{u}^2)(-A_-)\Big[ik_+R_0\,e^{ik_+x'}-ik_-\,e^{-ik_-x'}\Big]+(c^2-\bar{u}^2)B_+\Big[ik_+\,e^{ik_+(x'-L)}-ik_-R_L\,e^{-ik_-(x'-L)}\Big]\Big\}+\\
&\delta'(x-x')\Big\{(c^2-\bar{u}^2)(-A_-)\Big[R_0\,e^{ik_+x'}+e^{-ik_-x'}\Big]+(c^2-\bar{u}^2)B_+\Big[e^{ik_+(x'-L)}+R_L\,e^{-ik_-(x'-L)}\Big]\Big\}=-\delta(x-x')
\end{aligned}
\tag{5.18}
$$

A number of terms cancel (this is indicated by the coloured lines) because k_+ and k_- satisfy

$$(c^2-\bar{u}^2)k_+^2+2\bar{u}\omega k_+-\omega^2=0\,, \tag{5.19a}$$

$$(c^2-\bar{u}^2)k_-^2-2\bar{u}\omega k_--\omega^2=0\,, \tag{5.19b}$$

(this is a consequence of (3) and (4)). The remaining part of (5.18) is

$$
\begin{aligned}
&\delta(x-x')\Big\{-2\bar{u}\,i\omega A_-\Big[R_0\,e^{ik_+x'}+e^{-ik_-x'}\Big]-(c^2-\bar{u}^2)A_-\Big[ik_+R_0\,e^{ik_+x'}-ik_-\,e^{-ik_-x'}\Big]+\\
&\qquad+2\bar{u}\,i\omega B_+\Big[e^{ik_+(x'-L)}+R_L\,e^{-ik_-(x'-L)}\Big]+(c^2-\bar{u}^2)B_+\Big[ik_+\,e^{ik_+(x'-L)}-ik_-R_L\,e^{-ik_-(x'-L)}\Big]\Big\}+\\
&\delta'(x-x')\Big\{-(c^2-\bar{u}^2)\,A_-\Big[R_0\,e^{ik_+x'}+e^{-ik_-x'}\Big]+(c^2-\bar{u}^2)B_+\Big[e^{ik_+(x'-L)}+R_L\,e^{-ik_-(x'-L)}\Big]\Big\}=-\delta(x-x')
\end{aligned}
\tag{5.20}
$$

We simplify the blue term (T_A) and the pink term (T_B) in (5.20). For T_A we get

$$
\begin{aligned}
T_A&=A_-\Big\{R_0\,e^{ik_+x'}\Big[-2\bar{u}\,i\omega-(c^2-\bar{u}^2)\,ik_+\Big]+e^{-ik_-x'}\Big[-2\bar{u}\,i\omega+(c^2-\bar{u}^2)\,ik_-\Big]\Big\}=\\
&=A_-\Big\{R_0\,e^{ik_+x'}\frac{1}{ik_+}\Big[2\bar{u}\omega k_++(c^2-\bar{u}^2)\,k_+^2\Big]+e^{-ik_-x'}\frac{1}{ik_-}\Big[2\bar{u}\omega k_--(c^2-\bar{u}^2)\,k_-^2\Big]\Big\}
\end{aligned}
\tag{5.21}
$$

With (5.19a, b), the green and the red terms reduce to $\pm\omega^2$, so

$$
\begin{aligned}
T_A&=A_-\Big\{R_0\,e^{ik_+x'}\frac{\omega^2}{ik_+}-e^{-ik_-x'}\frac{\omega^2}{ik_-}\Big\}=\\
&=A_-\Big\{R_0\,e^{ik_+x'}(-i\omega)(c+\bar{u})-e^{-ik_-x'}(-i\omega)(c-\bar{u})\Big\}.
\end{aligned}
\tag{5.22}
$$

In the last step, (4a,b) were used. For T_B in (5.20) we get

$$T_B = B_+\left\{e^{ik_+(x'-L)}\left[2\bar{u}\,i\omega + (c^2-\bar{u}^2)\,ik_+\right] + R_L\,e^{-ik_-(x'-L)}\left[2\bar{u}\,i\omega + (c^2-\bar{u}^2)(-ik_-)\right]\right\} =$$

$$= B_+\left\{e^{ik_+(x'-L)}\frac{1}{ik_+}\left[-2\bar{u}\omega k_+ - (c^2-\bar{u}^2)\,k_+^2\right] + R_L\,e^{-ik_-(x'-L)}\frac{1}{ik_-}\left[-2\bar{u}\omega k_- + (c^2-\bar{u}^2)\,k_-^2\right]\right\}$$

(5.23)

Again, with (5.19a, b), the green and the red terms reduce to $\mp\omega^2$, so

$$T_B = B_+\left\{-e^{ik_+(x'-L)}\frac{\omega^2}{ik_+} + R_L\,e^{-ik_-(x'-L)}\frac{\omega^2}{ik_-}\right\} =$$

$$= B_+\left\{-e^{ik_+(x'-L)}(-i\omega)(c+\bar{u}) + R_L\,e^{-ik_-(x'-L)}(-i\omega)(c-\bar{u})\right\}. \qquad (5.24)$$

Then (5.20) becomes

$$\delta(x-x')\Big\{A_-\left[R_0\,e^{ik_+x'}(-i\omega)(c+\bar{u}) - e^{-ik_-x'}(-i\omega)(c-\bar{u})\right] +$$
$$+ B_+\left[-e^{ik_+(x'-L)}(-i\omega)(c+\bar{u}) + R_L\,e^{-ik_-(x'-L)}(-i\omega)(c-\bar{u})\right]\Big\} +$$
$$\delta'(x-x')\Big\{-A_-(c^2-\bar{u}^2)\left[R_0\,e^{ik_+x'} + e^{-ik_-x'}\right] + B_+(c^2-\bar{u}^2)\left[e^{ik_+(x'-L)} + R_L\,e^{-ik_-(x'-L)}\right]\Big\} = -\delta(x-x')$$

(5.25)

By comparing the coefficients of $\delta(x-x')$ and $\delta'(x-x')$, we obtain the following two equations

$$A_-(-i\omega)\left[R_0\,e^{ik_+x'}(c+\bar{u}) - e^{-ik_-x'}(c-\bar{u})\right] + B_+(-i\omega)\left[-e^{ik_+(x'-L)}(c+\bar{u}) + R_L\,e^{-ik_-(x'-L)}(c-\bar{u})\right] = -1$$

$$A_-(c^2-\bar{u}^2)\left[-R_0\,e^{ik_+x'} - e^{-ik_-x'}\right] + B_+(c^2-\bar{u}^2)\left[e^{ik_+(x'-L)} + R_L\,e^{-ik_-(x'-L)}\right] = 0\,. \qquad (5.26a,b)$$

This is a 2×2 set of linear equations for A_- and B_+, with determinant

$$\det = \begin{vmatrix} (-i\omega)\left[R_0\,e^{ik_+x'}(c+\bar{u}) - e^{-ik_-x'}(c-\bar{u})\right] & (-i\omega)\left[-e^{ik_+(x'-L)}(c+\bar{u}) + R_L\,e^{-ik_-(x'-L)}(c-\bar{u})\right] \\ -R_0\,e^{ik_+x'} - e^{-ik_-x'} & e^{ik_+(x'-L)} + R_L\,e^{-ik_-(x'-L)} \end{vmatrix}$$

(5.27)

We abbreviate the matrix elements so that the determinant can be written as

$$\det = \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix} = d_{11}d_{22} - d_{12}d_{21} \qquad (5.28)$$

and calculated.

$$d_{11}d_{22} = (-\mathrm{i}\omega)\left[R_0\,\mathrm{e}^{\mathrm{i}k_+x'}(c+\bar{u}) - \mathrm{e}^{-\mathrm{i}k_-x'}(c-\bar{u})\right]\left(\mathrm{e}^{\mathrm{i}k_+(x'-L)} + R_L\,\mathrm{e}^{-\mathrm{i}k_-(x'-L)}\right) =$$

$$= (-\mathrm{i}\omega)\left[\cancel{R_0\,\mathrm{e}^{\mathrm{i}k_+(2x'-L)}}(c+\bar{u}) - \cancel{R_L\,\mathrm{e}^{-\mathrm{i}k_-(2x'-L)}}(c-\bar{u}) + R_0R_L\,\mathrm{e}^{\mathrm{i}k_+x'}\mathrm{e}^{-\mathrm{i}k_-(x'-L)}(c+\bar{u}) - \mathrm{e}^{-\mathrm{i}k_-x'}\mathrm{e}^{\mathrm{i}k_+(x'-L)}(c-\bar{u})\right] \tag{5.29a}$$

$$d_{12}d_{21} = (-\mathrm{i}\omega)\left[-\mathrm{e}^{\mathrm{i}k_+(x'-L)}(c+\bar{u}) + R_L\,\mathrm{e}^{-\mathrm{i}k_-(x'-L)}(c-\bar{u})\right]\left(-R_0\,\mathrm{e}^{\mathrm{i}k_+x'} - \mathrm{e}^{-\mathrm{i}k_-x'}\right) =$$

$$= (-\mathrm{i}\omega)\left[\cancel{R_0\,\mathrm{e}^{\mathrm{i}k_+(2x'-L)}}(c+\bar{u}) - \cancel{R_L\,\mathrm{e}^{-\mathrm{i}k_-(2x'-L)}}(c-\bar{u}) - R_0R_L\,\mathrm{e}^{\mathrm{i}k_+x'}\mathrm{e}^{-\mathrm{i}k_-(x'-L)}(c-\bar{u}) + \mathrm{e}^{-\mathrm{i}k_-x'}\mathrm{e}^{\mathrm{i}k_+(x'-L)}(c+\bar{u})\right] \tag{5.29b}$$

The terms marked by the blue and green lines cancel, and $d_{11}d_{22} - d_{12}d_{21}$ becomes

$$\begin{aligned}
d_{11}d_{22} - d_{12}d_{21} &= (-\mathrm{i}\omega)\Big[R_0R_L\,\mathrm{e}^{\mathrm{i}k_+x'}\mathrm{e}^{-\mathrm{i}k_-(x'-L)}(c+\bar{u}) - \mathrm{e}^{-\mathrm{i}k_-x'}\mathrm{e}^{\mathrm{i}k_+(x'-L)}(c-\bar{u}) + \\
&\qquad + R_0R_L\,\mathrm{e}^{\mathrm{i}k_+x'}\mathrm{e}^{-\mathrm{i}k_-(x'-L)}(c-\bar{u}) - \mathrm{e}^{-\mathrm{i}k_-x'}\mathrm{e}^{\mathrm{i}k_+(x'-L)}(c+\bar{u})\Big] = \\
&= (-\mathrm{i}\omega)\left\{R_0R_L\,\mathrm{e}^{\mathrm{i}k_+x'}\mathrm{e}^{-\mathrm{i}k_-(x'-L)}\left[(c+\bar{u})+(c-\bar{u})\right] + \mathrm{e}^{-\mathrm{i}k_-x'}\mathrm{e}^{\mathrm{i}k_+(x'-L)}\left[-(c-\bar{u})-(c+\bar{u})\right]\right\} = \\
&= (-\mathrm{i}\omega)\left\{R_0R_L\,\mathrm{e}^{\mathrm{i}k_+x'}\mathrm{e}^{-\mathrm{i}k_-x'}\mathrm{e}^{\mathrm{i}k_-L}(2c) + \mathrm{e}^{-\mathrm{i}k_-x'}\mathrm{e}^{\mathrm{i}k_+x'}\mathrm{e}^{-\mathrm{i}k_+L}(-2c)\right\} = \\
&= (-\mathrm{i}\omega)\left\{2c\,\mathrm{e}^{-\mathrm{i}k_-x'}\mathrm{e}^{\mathrm{i}k_+x'}\mathrm{e}^{-\mathrm{i}k_+L}\left[R_0R_L\,\mathrm{e}^{\mathrm{i}k_+L}\mathrm{e}^{\mathrm{i}k_-L} - 1\right]\right\} = \\
&= (-\mathrm{i}\omega)\,2c\,\mathrm{e}^{\mathrm{i}(k_+-k_-)x'}\mathrm{e}^{-\mathrm{i}k_+L}\underbrace{\left(R_0R_L\mathrm{e}^{\mathrm{i}\omega\frac{2cL}{c^2-\bar{u}^2}} - 1\right)}_{=F(\omega)\ \text{(see (5.7b))}}.
\end{aligned} \tag{5.30}$$

In the last step, we used (5.6) to combine the exponential functions in the square brackets.

The linear equations (5.26) can now be solved for A_- and B_+.

$$A_- = \frac{\det_A}{\det},\quad B_+ = \frac{\det_B}{\det}, \tag{5.31a, b}$$

where

$$\det_A = \begin{vmatrix} -1 & d_{12} \\ 0 & d_{22} \end{vmatrix} = -d_{22} = -\mathrm{e}^{\mathrm{i}k_+(x'-L)} - R_L\,\mathrm{e}^{-\mathrm{i}k_-(x'-L)} \tag{5.32a}$$

$$\det_B = \begin{vmatrix} d_{11} & -1 \\ d_{21} & 0 \end{vmatrix} = d_{21} = -R_0\,\mathrm{e}^{\mathrm{i}k_+x'} - \mathrm{e}^{-\mathrm{i}k_-x'} \tag{5.32b}$$

For A_- we get

$$A_- = \frac{-e^{ik_+(x'-L)} - R_L e^{-ik_-(x'-L)}}{(-i\omega)2c\, e^{i(k_+-k_-)x'} e^{-ik_+L} F(\omega)} =$$

$$= \frac{-1}{2cF(\omega)(-i\omega)} e^{-i(k_+-k_-)x'} e^{ik_+L} \underbrace{\left[e^{ik_+(x'-L)} + R_L e^{-ik_-(x'-L)}\right]}_{=b(x',\omega)\quad \text{(see (5.14))}}. \tag{5.33a}$$

For B_+ we get

$$B_+ = \frac{-R_0 e^{ik_+x'} - e^{-ik_-x'}}{(-i\omega)2c\, e^{i(k_+-k_-)x'} e^{-ik_+L} F(\omega)} =$$

$$= \frac{-1}{2cF(\omega)(-i\omega)} e^{-i(k_+-k_-)x'} e^{ik_+L} \underbrace{\left[R_0 e^{ik_+x'} + e^{-ik_-x'}\right]}_{=a(x',\omega)\quad \text{(see (5.14))}}. \tag{5.33b}$$

Altogether, we get with (5.14) for the Green's function

$$\hat{g}(x,x',\omega) = \begin{cases} \dfrac{-1}{2cF(\omega)(-i\omega)} e^{-i(k_+-k_-)x'} e^{ik_+L}\, b(x',\omega)a(x,\omega) & \text{for } x < x' \\ \dfrac{-1}{2cF(\omega)(-i\omega)} e^{-i(k_+-k_-)x'} e^{ik_+L}\, a(x',\omega)b(x,\omega) & \text{for } x > x' \end{cases} \tag{5.34}$$

with

$$F(\omega) = -1 + R_0 R_L e^{i\omega \frac{2cL}{c^2-\bar{u}^2}}, \tag{5.35a}$$

$$a(x,\omega) = R_0 e^{ik_+x} + e^{-ik_-x}, \tag{5.35b}$$

$$b(x,\omega) = e^{ik_+(x-L)} + R_L e^{-ik_-(x-L)}. \tag{5.35c}$$

Due to the pink term in (5.34), this Green's function does not satisfy reciprocity: $\hat{g}(x,x',\omega) \neq \hat{g}(x',x,\omega)$, unless $k_+ - k_- = 0$, which is valid only for the case with zero mean flow ($\bar{u} = 0$). The lack of reciprocity may be surprising, but it is plausible for a flow-duct: if the source is upstream of the receiver, the emitted sound wave travels with the flow; however, if the source and receiver are swapped over, the wave has to travel against the flow.

Appendix 5.3: Calculation of the time-domain Green's function

The time-domain Green's function $g(x,x',t-t')$ is calculated from

$$g(x,x',t-t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(x,x',\omega) e^{-i\omega(t-t')}\, d\omega . \quad (5.36)$$

It must satisfy causality, i.e.

$$g(x,x',t-t') = 0 \quad \text{for} \quad t-t' < 0 . \quad (5.37)$$

For $t-t' > 0$, g must be a superposition of modes, and we now proceed to show this by determining the integral in (5.36) with the residue theorem.

Equation (5.25) shows that $\hat{g}(x,x',\omega)$ has singularities at $\omega = 0$ and $\omega = \omega_n$ (because $F(\omega_n) = 0$), so the integrand in (5.36) has singularities at the same frequencies. Their position in the complex plane is shown in figure 5.2.

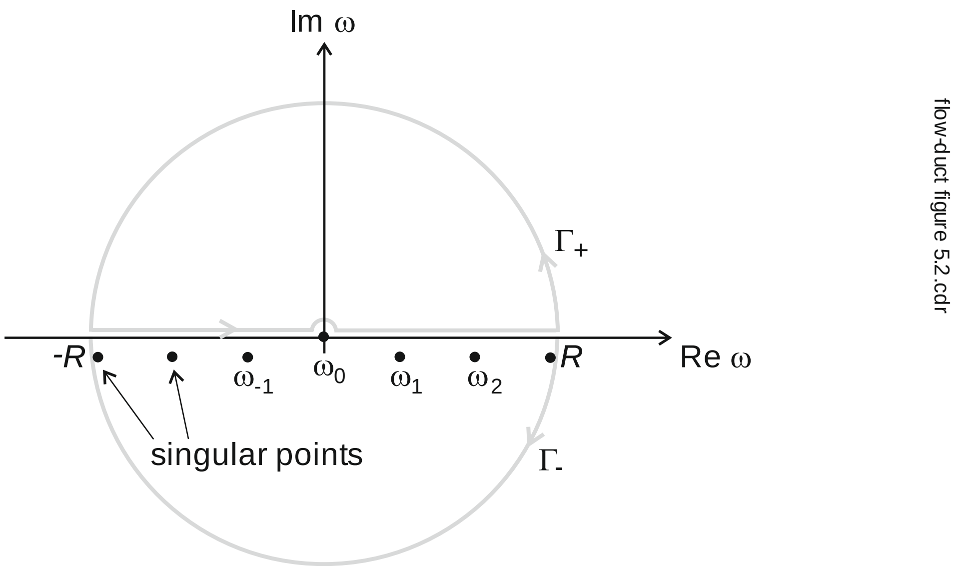

Figure 5.2. Singular points in the complex ω-plane

For the integration path, we choose the closed curve composed of the real axis and the semi-circular arc Γ_- in the lower half-plane; this curve encloses all singular points.

Application of the residue theorem gives (the path is traversed in the negative direction, hence the minus sign in $-2\pi i$)

$$\int_{-\infty}^{\infty} \hat{g}(x,x',\omega)\mathrm{e}^{-\mathrm{i}\omega(t-t')}\,\mathrm{d}\omega = -2\pi\mathrm{i}\sum_{n=-\infty}^{\infty} \mathrm{Res}_{\omega_n}\left[\hat{g}(x,x',\omega)\mathrm{e}^{-\mathrm{i}\omega(t-t')}\right] - \underbrace{-\lim_{R\to\infty}\int_{\Gamma_-} \hat{g}(x,x',\omega)\mathrm{e}^{-\mathrm{i}\omega(t-t')}\,\mathrm{d}\omega}_{=0}. \tag{5.38}$$

The integral along Γ_- is zero for $t-t'>0$, because the exponential function tends to zero as the radius of the semicircle tends to infinity,

$$\mathrm{e}^{-\mathrm{i}\omega(t-t')} = \mathrm{e}^{-\mathrm{i}(\omega_r+\mathrm{i}\omega_i)(t-t')} = \underbrace{\mathrm{e}^{-\mathrm{i}\omega_r(t-t')}}_{\text{bounded}}\,\mathrm{e}^{\omega_i(t-t')} \to 0 \quad \text{as} \quad \omega_i \to -\infty. \tag{5.39}$$

It now remains to calculate the residues of $\hat{g}(x,x',\omega)\mathrm{e}^{-\mathrm{i}\omega(t-t')}$. We introduce the abbreviation

$$\psi(x,\omega) = -\frac{1}{c}\mathrm{e}^{-\mathrm{i}(k_+-k_-)x}\mathrm{e}^{-\mathrm{i}k_+L} = -\frac{1}{c}\mathrm{e}^{\mathrm{i}\omega x\frac{2\bar{u}}{c^2-\bar{u}^2}}\,\mathrm{e}^{-\mathrm{i}\frac{\omega}{c+\bar{u}}L}, \tag{5.40}$$

and use (5.34) to write the residue term as

$$\hat{g}(x,x',\omega)\mathrm{e}^{-\mathrm{i}\omega(t-t')} = \frac{\mathrm{i}}{2\omega F(\omega)}\begin{cases}\psi(x',\omega)b(x',\omega)a(x,\omega)\mathrm{e}^{-\mathrm{i}\omega(t-t')} & \text{for} \quad x<x' \\ \psi(x',\omega)a(x',\omega)b(x,\omega)\mathrm{e}^{-\mathrm{i}\omega(t-t')} & \text{for} \quad x>x'\end{cases} \tag{5.41}$$

This is a quotient, and the following general formula for the calculation of a residue can be applied,

$$\mathrm{Res}_{\omega_n}\left[\frac{P(\omega)}{Q(\omega)}\right] = \frac{P(\omega_n)}{Q'(\omega_n)}. \tag{5.42}$$

Here, we have

$$P(\omega) = \mathrm{i}\begin{cases}\psi(x',\omega)b(x',\omega)a(x,\omega)\mathrm{e}^{-\mathrm{i}\omega(t-t')} & \text{for} \quad x<x' \\ \psi(x',\omega)a(x',\omega)b(x,\omega)\mathrm{e}^{-\mathrm{i}\omega(t-t')} & \text{for} \quad x>x'\end{cases} \tag{5.43a}$$

$$Q(\omega) = 2\omega F(\omega) \tag{5.43b}$$

$$Q'(\omega_n) = \left.\frac{\mathrm{d}Q}{\mathrm{d}\omega}\right|_{\omega_n} = \left[2F(\omega)+2\omega F'(\omega)\right]_{\omega_n} = 2\omega_n F'(\omega_n) \tag{5.43c}$$

This gives for the residue in (5.38)

$$\mathrm{Res}_{\omega_n}[\ldots] = \frac{\mathrm{i}\psi(x',\omega_n)\mathrm{e}^{-\mathrm{i}\omega_n(t-t')}}{2\omega_n F'(\omega_n)}\begin{cases}b(x',\omega_n)a(x,\omega_n) & \text{for} \quad x<x' \\ a(x',\omega_n)b(x,\omega_n) & \text{for} \quad x>x'\end{cases} \tag{5.44}$$

We insert this into (5.38) and subsequently into (5.36), to get the time-domain Green's function

$$g(x,x',t-t') = \frac{1}{2\pi}(-2\pi\mathrm{i}) \sum_{n=-\infty}^{\infty} \mathrm{Res}_{\omega_n}[...] =$$

$$= -\mathrm{i} \sum_{n=-\infty}^{\infty} \frac{\mathrm{i}\psi(x',\omega_n)\mathrm{e}^{-\mathrm{i}\omega_n(t-t')}}{2\omega_n F'(\omega_n)} \begin{cases} b(x',\omega_n)a(x,\omega_n) & \text{for } x < x' \\ a(x',\omega_n)b(x,\omega_n) & \text{for } x > x' \end{cases}$$

$$= \sum_{n=-\infty}^{\infty} \frac{g_n(x,x',\omega_n)}{2\omega_n F'(\omega_n)} \mathrm{e}^{-\mathrm{i}\omega_n(t-t')}, \tag{5.45}$$

with

$$g_n(x,x',\omega_n) = \begin{cases} \psi(x',\omega_n)b(x',\omega_n)a(x,\omega_n) & \text{for } x < x' \\ \psi(x',\omega_n)a(x',\omega_n)b(x,\omega_n) & \text{for } x > x' \end{cases} \tag{5.46}$$

The negative mode numbers in (5.45) can be eliminated. To this end, we split the sum into three parts,

$$\sum_{n=-\infty}^{\infty} = \sum_{n=1}^{\infty} + \sum_{n=-1}^{-\infty} + \sum_{n=0}^{0} . \tag{5.47}$$

We rewrite the sum over the negative mode numbers,

$$\sum_{n=-1}^{-\infty} \frac{g_n(x,x',\omega_n)}{2\omega_n F'(\omega_n)} \mathrm{e}^{-\mathrm{i}\omega_n(t-t')} = \sum_{n=1}^{\infty} \frac{g_n(x,x',\omega_{-n})}{2\omega_{-n} F'(\omega_{-n})} \mathrm{e}^{-\mathrm{i}\omega_{-n}(t-t')} . \tag{5.48}$$

As a consequence of

$$\omega_{-n} = -\omega_n^* \tag{5.49}$$

(see [Heckl 2009]), we can use

$$g_n(x,x',\omega_{-n}) = [g_n(x,x',\omega_n)]^* \quad \text{(see Appendix 5.4)}, \tag{5.50a}$$

$$F'(\omega_{-n}) = -[F'(\omega_n)]^* \quad \text{(see Appendix 5.4)}, \tag{5.50b}$$

$$\mathrm{e}^{-\mathrm{i}\omega_{-n}(t-t')} = \left[\mathrm{e}^{-\mathrm{i}\omega_n(t-t')}\right]^*, \tag{5.50c}$$

and rewrite (5.48),

$$\sum_{n=-1}^{-\infty} \frac{g_n(x,x',\omega_n)}{2\omega_n F'(\omega_n)} \mathrm{e}^{-\mathrm{i}\omega_n(t-t')} = \left[\sum_{n=1}^{\infty} \frac{g_n(x,x',\omega_n)}{2\omega_n F'(\omega_n)} \mathrm{e}^{-\mathrm{i}\omega_n(t-t')}\right]^* . \tag{5.51}$$

In other words, the sum over the negative mode numbers in (5.47) is the complex conjugate of the sum over positive mode numbers.

The $n = 0$ term is zero (see Appendix 5.5). We then get

$$g(x,x',t-t') = \sum_{n=1}^{\infty} \ \dots + \left[\sum_{n=1}^{\infty} \ \dots \right]^* = 2\mathrm{Re}\left[\sum_{n=1}^{\infty} \ \dots \right] = \sum_{n=1}^{\infty} \mathrm{Re}\left[\frac{g_n(x,x',\omega_n)}{\omega_n F'(\omega_n)} \mathrm{e}^{-\mathrm{i}\omega_n(t-t')}\right]. \tag{5.52}$$

Appendix 5.4: Negative frequencies

$$\omega_{-n} = -\omega_n{}^* \tag{5.53}$$

Reflection coefficients:

$$R_0(\omega_{-n}) \underset{\substack{\uparrow \\ \text{see [Heckl 2009]}}}{=} R_0(-\omega_n{}^*) = [R_0(\omega_n{}^*)]^* \underset{\substack{\uparrow \\ \text{ignore imag of } \omega_n}}{=} [R_0(\omega_n)]^* \tag{5.54}$$

$$R_L(\omega_{-n}) = \ldots = [R_L(\omega_n)]^* \tag{5.55}$$

Function $F(\omega)$:

$$F(\omega) = R_0 R_L \mathrm{e}^{\mathrm{i}\omega \frac{2cL}{c^2-\bar{u}^2}} - 1 \tag{5.56}$$

$$F'(\omega) = \mathrm{i}\frac{2cL}{c^2-\bar{u}^2} R_0 R_L \mathrm{e}^{\mathrm{i}\omega \frac{2cL}{c^2-\bar{u}^2}} \tag{5.57}$$

$$F'(\omega_{-n}) = F'(-\omega_n{}^*) = \mathrm{i}\frac{2cL}{c^2-\bar{u}^2} R_0{}^* \; R_L{}^* \, \mathrm{e}^{-\mathrm{i}\omega_n{}^* \frac{2cL}{c^2-\bar{u}^2}} = -\left[\mathrm{i}\frac{2cL}{c^2-\bar{u}^2} R_0 R_L \mathrm{e}^{\mathrm{i}\omega_n \frac{2cL}{c^2-\bar{u}^2}}\right]^* =$$

$$= -\left[F'(\omega_n)\right]^* \tag{5.58}$$

Wave numbers:

$$k_\pm(\omega_{-n}) = k_\pm(-\omega_n{}^*) = \frac{-\omega_n{}^*}{c \pm \bar{u}} = -[k_\pm(\omega_n)]^* \tag{5.59}$$

Function $g_n(x, x', \omega_n)$:

$$a(x, \omega_{-n}) = R_0{}^* \; \mathrm{e}^{-\mathrm{i}k_+{}^* x} + \mathrm{e}^{\mathrm{i}k_-{}^* x} = \left[R_0 \, \mathrm{e}^{\mathrm{i}k_+ x} + \mathrm{e}^{-\mathrm{i}k_- x}\right]^* = [a(x, \omega_n)]^* \tag{5.60}$$

$$b(x, \omega_{-n}) = \mathrm{e}^{-\mathrm{i}k_+{}^*(x-L)} + R_L{}^* \; \mathrm{e}^{\mathrm{i}k_-{}^*(x-L)} = \left[\mathrm{e}^{\mathrm{i}k_+(x-L)} + R_L \, \mathrm{e}^{-\mathrm{i}k_-(x-L)}\right]^* = [b(x, \omega_n)]^* \tag{5.61}$$

$$\psi(x, \omega_{-n}) = c \, \mathrm{e}^{-\mathrm{i}(k_+{}^* - k_-{}^*)x} \mathrm{e}^{-\mathrm{i}k_+{}^* L} = c\left[\mathrm{e}^{-\mathrm{i}(k_+ - k_-)x} \mathrm{e}^{-\mathrm{i}k_+ L}\right]^* = [\psi(x, \omega_n)]^* \tag{5.62}$$

Hence

$$g_n(x, x', \omega_{-n}) = \begin{cases} \psi^*(x, \omega_n) b^*(x', \omega) a^*(x, \omega) & \text{for } \; x < x' \\ \psi^*(x, \omega_n) a^*(x', \omega) b^*(x, \omega) & \text{for } \; x > x' \end{cases}$$

$$= [g_n(x, x', \omega_n)]^* \tag{5.63}$$

Appendix 5.5: Residue at $\omega_0 = 0$

The calculation of the residue of $\hat{g}(x,x',\omega)e^{-i\omega(t-t')}$ in Appendix 5.3 is valid for $\omega \neq 0$, but not for $\omega = 0$. In this additional appendix, we calculate the residue for $\omega = 0$; this corresponds to $n = 0$ in our notation, i.e. $\omega_0 = 0$. We do this by expanding $\hat{g}(x,x',\omega)e^{-i\omega(t-t')}$ into a Laurent series about $\omega = 0$; the coefficient of ω^{-1} will then give the required residue.

From (5.41), we know

$$\hat{g}(x,x',\omega)e^{-i\omega(t-t')} = \frac{i\psi(x',\omega)}{2\omega F(\omega)} \begin{cases} b(x',\omega)a(x,\omega)e^{-i\omega(t-t')} & \text{for } x < x' \\ a(x',\omega)b(x,\omega)e^{-i\omega(t-t')} & \text{for } x > x' \end{cases} \tag{5.64}$$

with

$$F(\omega) = -1 + R_0 R_L e^{i\omega\frac{2cL}{c^2-\bar{u}^2}}, \tag{5.65a}$$

$$a(x,\omega) = R_0\, e^{i\frac{\omega x}{c+\bar{u}}} + e^{-i\frac{\omega x}{c-\bar{u}}}, \tag{5.65b}$$

$$b(x,\omega) = e^{i\frac{\omega(x-L)}{c+\bar{u}}} + R_L\, e^{-i\frac{\omega(x-L)}{c-\bar{u}}}, \tag{5.65c}$$

$$\psi(x,\omega) = -\frac{1}{c} e^{i\omega x\frac{2\bar{u}}{c^2-\bar{u}^2}}\, e^{-i\frac{\omega}{c+\bar{u}}L}. \tag{5.65d}$$

We construct the Laurent series by expanding the individual terms in (5.65) into Taylor series (in terms of ω) and then insert these Taylor series into (5.64).

It is important to take into account that the reflection coefficients R_0 and R_L also depend on ω, so we need an approximation for them that is valid for small ω. To this end, we consider the reflection coefficient for an open tube end (derived by [Levine and Schwinger 1948])

$$R = -\frac{1-[\frac{1}{4}(\frac{\omega r}{c})^2 - i\frac{\omega r}{c}0.6133]}{1+[\frac{1}{4}(\frac{\omega r}{c})^2 - i\frac{\omega r}{c}0.6133]}, \tag{5.66}$$

where r is the radius of the tube. For small ω, the term in the square brackets is small, so R can be approximated as follows.

$$R=-\frac{1-[\ldots]}{1+[\ldots]}\approx-\left(1-[\ldots]\right)^2\approx-\left\{1-\omega\underbrace{(-\mathrm{i}\frac{2r}{c}0.6133)}_{=\alpha}\right\}=-(1-\alpha\omega)\,. \tag{5.67}$$

In the last two steps of (5.67), we only included the linear ω-term, but not the ω^2 - term; also we introduced the abbreviation α for the constant term.

We can now describe the tube ends at $x=0$ and $x=L$ by the approximated reflection coefficients

$$R_0=-1+\alpha_0\omega\,, \tag{5.68a}$$

$$R_L=-1+\alpha_L\omega\,, \tag{5.68b}$$

for small frequencies; α_0 and α_L are (complex) constants. This allows us to approximate the expressions in (5.65) as follows.

$$\begin{aligned}F(\omega)&=-1+(-1+\alpha_0\omega)(-1+\alpha_L\omega)\left(1+\omega\frac{2\mathrm{i}cL}{c^2-\bar{u}^2}\right)+O(\omega^2)=\\&=-1+(1-\alpha_0\omega-\alpha_L\omega+\alpha_0\alpha_L\omega^2)\left(1+\omega\frac{2\mathrm{i}cL}{c^2-\bar{u}^2}\right)+O(\omega^2)=\\&=-1+(1-\alpha_0\omega-\alpha_L\omega)+\omega\frac{2\mathrm{i}cL}{c^2-\bar{u}^2}+O(\omega^2)=\\&=\omega\left(\frac{2\mathrm{i}cL}{c^2-\bar{u}^2}-\alpha_0-\alpha_L\right)+O(\omega^2)\end{aligned} \tag{5.69}$$

$$\begin{aligned}a(x,\omega)&=(-1+\alpha_0\omega)\left(1+\frac{\mathrm{i}\omega x}{c+\bar{u}}\right)+\left(1-\frac{\mathrm{i}\omega x}{c-\bar{u}}\right)+O(\omega^2)=\\&=-1+\alpha_0\omega-\frac{\mathrm{i}\omega x}{c+\bar{u}}+1-\frac{\mathrm{i}\omega x}{c-\bar{u}}+O(\omega^2)=\\&=\omega\left(\alpha_0-\frac{\mathrm{i}x}{c+\bar{u}}-\frac{\mathrm{i}x}{c-\bar{u}}\right)+O(\omega^2)\end{aligned} \tag{5.70}$$

$$\begin{aligned}b(x,\omega)&=\left(1+\frac{\mathrm{i}\omega(x-L)}{c+\bar{u}}\right)+(-1+\alpha_L\omega)\left(1-\frac{\mathrm{i}\omega(x-L)}{c-\bar{u}}\right)+O(\omega^2)=\\&=1+\frac{\mathrm{i}\omega(x-L)}{c+\bar{u}}-1+\frac{\mathrm{i}\omega(x-L)}{c-\bar{u}}+\alpha_L\omega+O(\omega^2)=\\&=\omega\left(\alpha_L+\frac{\mathrm{i}(x-L)}{c+\bar{u}}+\frac{\mathrm{i}(x-L)}{c-\bar{u}}\right)+O(\omega^2)\end{aligned} \tag{5.71}$$

$$\psi(x,\omega) = -\frac{1}{c}\left(1-\frac{2\mathrm{i}\omega x\bar{u}}{c^2-\bar{u}^2}\right)\left(1-\mathrm{i}\frac{\omega L}{c+\bar{u}}\right)+O(\omega^2) =$$

$$= -\frac{1}{c}\left[1-\omega\left(\frac{2\mathrm{i}x\bar{u}}{c^2-\bar{u}^2}+\frac{\mathrm{i}L}{c+\bar{u}}\right)\right]+O(\omega^2) \tag{5.72}$$

$$\mathrm{e}^{-\mathrm{i}\omega(t-t')} = 1-\mathrm{i}\omega(t-t')+O(\omega^2) \tag{5.73}$$

The expansions (5.69) to (5.73) are now substituted into (5.64), omitting the terms $O(\omega^2)$. For $x < x'$, we get

$$\hat{g}(x,x',\omega)\mathrm{e}^{-\mathrm{i}\omega(t-t')} =$$

$$= \frac{-\mathrm{i}}{2c}\frac{\left[1-\omega\left(\frac{2\mathrm{i}x'\bar{u}}{c^2-\bar{u}^2}+\frac{\mathrm{i}L}{c+\bar{u}}\right)\right]\omega\left(\alpha_L+\frac{\mathrm{i}(x'-L)}{c+\bar{u}}+\frac{\mathrm{i}(x'-L)}{c-\bar{u}}\right)\omega\left(\alpha_0-\frac{\mathrm{i}x}{c+\bar{u}}-\frac{\mathrm{i}x}{c-\bar{u}}\right)\left[1-\mathrm{i}\omega(t-t')\right]}{\omega\ \omega\left(\frac{2\mathrm{i}cL}{c^2-\bar{u}^2}-\alpha_0-\alpha_L\right)}$$

(5.74)

The two factors ω in the denominator cancel with those in the numerator, leaving a series without negative powers of ω on the right hand side of (5.74). This series is the required Laurent series; we can therefore conclude that the coefficient of ω^{-1} is zero. Hence the residue at $\omega_0 = 0$ is also zero.

The same result is obtained for $x > x'$.

I hope that the same result would be obtained for closed tube ends, where $R_0 = R_L = +1$, but I have not had the time to check this.

Appendix for Step 6: Determine the adjoint of $\hat{g}(x,x',\omega)$

The frequency-domain Green's function $\hat{g}(x,x',\omega)$ satisfies

$$\omega^2 \hat{g}(x,x',\omega) + 2\bar{u}\,\mathrm{i}\omega \frac{\partial \hat{g}}{\partial x} + \left(c^2 - \bar{u}^2\right) \frac{\partial^2 \hat{g}}{\partial x^2} = -\delta(x-x') \tag{6.1}$$

Multiply (6.1) by the test function $\hat{G}(x,x^*,\omega)$ (yet to be defined) and integrate

$\int_{x=0}^{L} \ldots \hat{G}(x,x^*,\omega) \ldots \mathrm{d}x$. This leads to

$$\omega^2 \int_{x=0}^{L} \hat{g}\hat{G}\,\mathrm{d}x + 2\bar{u}\,\mathrm{i}\omega \underbrace{\int_{x=0}^{L} \frac{\partial \hat{g}}{\partial x}\hat{G}\,\mathrm{d}x}_{=I_1} + (c^2-\bar{u}^2) \underbrace{\int_{x=0}^{L} \frac{\partial^2 \hat{g}}{\partial x^2}\hat{G}\,\mathrm{d}x}_{=I_2} = \underbrace{-\int_{x=0}^{L} \delta(x-x')\hat{G}(x,x^*,\omega)\,\mathrm{d}x}_{=-\hat{G}(x',x^*,\omega)} \tag{6.2}$$

$$I_1 = \int_{x=0}^{L} \frac{\partial \hat{g}}{\partial x}\hat{G}\,\mathrm{d}x = \left[\hat{g}\hat{G}\right]_{x=0}^{L} - \int_{x=0}^{L} \hat{g}\frac{\partial \hat{G}}{\partial x}\,\mathrm{d}x \tag{6.3}$$

$$I_2 = \int_{x=0}^{L} \frac{\partial^2 \hat{g}}{\partial x^2}\hat{G}\,\mathrm{d}x = \left[\frac{\partial \hat{g}}{\partial x}\hat{G}\right]_{x=0}^{L} - \int_{x=0}^{L} \frac{\partial \hat{g}}{\partial x}\frac{\partial \hat{G}}{\partial x}\,\mathrm{d}x =$$

$$= \left[\frac{\partial \hat{g}}{\partial x}\hat{G}\right]_{x=0}^{L} - \left\{\left[\hat{g}\frac{\partial \hat{G}}{\partial x}\right]_{x=0}^{L} - \int_{x=0}^{L} \hat{g}\frac{\partial^2 \hat{G}}{\partial x^2}\,\mathrm{d}x\right\} =$$

$$= \left[\hat{G}\frac{\partial \hat{g}}{\partial x} - \hat{g}\frac{\partial \hat{G}}{\partial x}\right]_{x=0}^{L} + \int_{x=0}^{L} \hat{g}\frac{\partial^2 \hat{G}}{\partial x^2}\,\mathrm{d}x \tag{6.4}$$

We substitute (6.3) and (6.4) into (6.2) and then obtain

$$\omega^2 \int_{x=0}^{L} \hat{g}\hat{G}\,\mathrm{d}x + 2\bar{u}\,\mathrm{i}\omega \left\{\left[\hat{g}\hat{G}\right]_{x=0}^{L} - \int_{x=0}^{L} \hat{g}\frac{\partial \hat{G}}{\partial x}\,\mathrm{d}x\right\} +$$

$$+(c^2-\bar{u}^2)\left\{\left[\hat{G}\frac{\partial \hat{g}}{\partial x} - \hat{g}\frac{\partial \hat{G}}{\partial x}\right]_{x=0}^{L} + \int_{x=0}^{L} \hat{g}\frac{\partial^2 \hat{G}}{\partial x^2}\,\mathrm{d}x\right\} = -\hat{G}(x',x^*,\omega) \tag{6.5}$$

or

$$\int_{x=0}^{L} \hat{g}\left[\omega^2\hat{G} - 2\bar{u}\,\mathrm{i}\omega\frac{\partial\hat{G}}{\partial x} + (c^2-\bar{u}^2)\frac{\partial^2\hat{G}}{\partial x^2}\right]\mathrm{d}x +$$

$$\underbrace{2\bar{u}\,\mathrm{i}\omega\Big[\hat{g}\hat{G}\Big]_{x=0}^{L} + (c^2-\bar{u}^2)\left[\hat{G}\frac{\partial\hat{g}}{\partial x} - \hat{g}\frac{\partial\hat{G}}{\partial x}\right]_{x=0}^{L}}_{=\mathrm{BT3}} = -\hat{G}(x',x^*,\omega)\,. \tag{6.6}$$

We define $\hat{G}$ by the following PDE:

$$\omega^2\hat{G}(x,x^*,\omega) - 2\bar{u}\,\mathrm{i}\omega\frac{\partial\hat{G}}{\partial x} + \left(c^2-\bar{u}^2\right)\frac{\partial^2\hat{G}}{\partial x^2} = -\delta(x-x^*)\,. \tag{6.7}$$

Then (6.6) leads to

$$\int_{x=0}^{L} \hat{g}(x,x',\omega)\delta(x-x^*)\,\mathrm{d}x - \mathrm{BT3} = \hat{G}(x',x^*,\omega)\,. \tag{6.8}$$

We will define $\hat{G}$ further by imposing boundary conditions, which make the boundary term BT3 disappear.

By comparing Eqs. (6.1) and (6.7), we notice that $\hat{g}$ and $\hat{G}$ satisfy very similar PDEs: they only differ by the sign of the mean flow velocity $\bar{u}$.

Given that $\hat{g}$ has known expressions on either side of the point x' (see Appendix for Step 5, Eq. (5.14))

$$\hat{g}(x,x',\omega) = \begin{cases} A_{-}(x',\omega)\,[R_0\,\mathrm{e}^{\frac{\mathrm{i}\omega}{c+\bar{u}}x} + \mathrm{e}^{-\frac{\mathrm{i}\omega}{c-\bar{u}}x}] & \text{near} \quad x=0 \\ B_{+}(x',\omega)\,[\mathrm{e}^{\frac{\mathrm{i}\omega}{c+\bar{u}}(x-L)} + R_L\,\mathrm{e}^{-\frac{\mathrm{i}\omega}{c-\bar{u}}(x-L)}] & \text{near} \quad x=L \end{cases} \tag{6.9}$$

we can construct a trial solution for $\hat{G}$ by changing the sign of $\bar{u}$ in the expressions above for $\hat{g}$, and allowing for different amplitudes ($\tilde{A}_{-}$ instead of A_{-}, and $\tilde{B}_{+}$ instead of B_{+}):

$$\hat{G}(x,x',\omega) = \begin{cases} \tilde{A}_{-}(x',\omega)\,[R_0\,\mathrm{e}^{\frac{\mathrm{i}\omega}{c-\bar{u}}x} + \mathrm{e}^{-\frac{\mathrm{i}\omega}{c+\bar{u}}x}] & \text{near} \quad x=0 \\ \tilde{B}_{+}(x',\omega)\,[\mathrm{e}^{\frac{\mathrm{i}\omega}{c-\bar{u}}(x-L)} + R_L\,\mathrm{e}^{-\frac{\mathrm{i}\omega}{c+\bar{u}}(x-L)}] & \text{near} \quad x=L \end{cases} \tag{6.10}$$

The derivatives of (6.9) and (6.10) are

$$\frac{\partial \hat{g}}{\partial x}(x,x',\omega)=\begin{cases} A_-(x',\omega)\mathrm{i}\omega[\frac{1}{c+\bar{u}}R_0\,\mathrm{e}^{\frac{\mathrm{i}\omega}{c+\bar{u}}x}-\frac{1}{c-\bar{u}}\mathrm{e}^{-\frac{\mathrm{i}\omega}{c-\bar{u}}x}] & \text{near} \quad x=0 \\ B_+(x',\omega)\,\mathrm{i}\omega[\frac{1}{c+\bar{u}}\mathrm{e}^{\frac{\mathrm{i}\omega}{c+\bar{u}}(x-L)}-\frac{1}{c-\bar{u}}R_L\mathrm{e}^{-\frac{\mathrm{i}\omega}{c-\bar{u}}(x-L)}] & \text{near} \quad x=L \end{cases} \tag{6.11}$$

and

$$\frac{\partial \hat{G}}{\partial x}(x,x',\omega)=\begin{cases} \tilde{A}_-(x',\omega)\mathrm{i}\omega[\frac{1}{c-\bar{u}}R_0\,\mathrm{e}^{\frac{\mathrm{i}\omega}{c-\bar{u}}x}-\frac{1}{c+\bar{u}}\mathrm{e}^{-\frac{\mathrm{i}\omega}{c+\bar{u}}x}] & \text{near} \quad x=0 \\ \tilde{B}_+(x',\omega)\,\mathrm{i}\omega[\frac{1}{c-\bar{u}}\mathrm{e}^{\frac{\mathrm{i}\omega}{c-\bar{u}}(x-L)}-\frac{1}{c+\bar{u}}R_L\mathrm{e}^{-\frac{\mathrm{i}\omega}{c+\bar{u}}(x-L)}] & \text{near} \quad x=L \end{cases} \tag{6.12}$$

We need to evaluate (6.9) – (6.12) at $x=0$ and $x=L$.

For $x=0$, we get

$$\hat{g}(0,x',\omega)=A_-(x',\omega)\left(R_0+1\right) \tag{6.13a}$$

$$\hat{G}(0,x',\omega)=\tilde{A}_-(x',\omega)\left(R_0+1\right) \tag{6.13b}$$

$$\left.\frac{\partial \hat{g}}{\partial x}\right|_{x=0}=A_-(x',\omega)\mathrm{i}\omega[\frac{1}{c+\bar{u}}R_0-\frac{1}{c-\bar{u}}] \tag{6.13c}$$

$$\left.\frac{\partial \hat{G}}{\partial x}\right|_{x=0}=\tilde{A}_-(x',\omega)\mathrm{i}\omega[\frac{1}{c-\bar{u}}R_0-\frac{1}{c+\bar{u}}] \tag{6.13d}$$

Then we get for the boundary at $x=0$ of the term BT3 in (6.6)

$$[\mathrm{BT3}]_{x=0} = 2\bar{u}\,\mathrm{i}\omega\left[\hat{g}\hat{G}\right]_{x=0} + (c^2-\bar{u}^2)\left[\hat{G}\frac{\partial\hat{g}}{\partial x} - \hat{g}\frac{\partial\hat{G}}{\partial x}\right]_{x=0} =$$

$$= 2\bar{u}\,\mathrm{i}\omega A\_(x',\omega)\tilde{A}\_(x',\omega)(R_0+1)^2 +$$

$$+(c^2-\bar{u}^2)A\_(x',\omega)\tilde{A}\_(x',\omega)\mathrm{i}\omega\left[(\frac{1}{c+\bar{u}}R_0 - \frac{1}{c-\bar{u}})(R_0+1) - (R_0+1)(\frac{1}{c-\bar{u}}R_0 - \frac{1}{c+\bar{u}})\right] =$$

$$= A\_(x',\omega)\tilde{A}\_(x',\omega)\mathrm{i}\omega(R_0+1)\left\{2\bar{u}(R_0+1) + (c^2-\bar{u}^2)\underbrace{\left[(\frac{1}{c+\bar{u}}R_0 - \frac{1}{c-\bar{u}}) - (\frac{1}{c-\bar{u}}R_0 - \frac{1}{c+\bar{u}})\right]}_{=(R_0+1)(\frac{1}{c+\bar{u}}-\frac{1}{c-\bar{u}}) = (R_0+1)\frac{-2\bar{u}}{c^2-\bar{u}^2}}\right\} = 0 \qquad (6.14)$$

The same calculation can be done for the end at $x = L$, and the result is also zero. Now $\hat{G}$ is completely defined: it has to satisfy the PDE (6.7) and the end conditions (6.10).

Appendix for Step 7: Find the relationship between $G(x',x,t',t)$ and $\hat{G}(x',x,\omega)$

$G(x',x,t',t)$ satisfies the PDE (see Eq. (7) in Step 4)

$$\frac{\partial^2 G}{\partial t'^2}+2\bar{u}\frac{\partial^2 G}{\partial t'\partial x'}-\left(c^2-\bar{u}^2\right)\frac{\partial^2 G}{\partial x'^2}=\delta(x'-x)\delta(t'-t)\,. \tag{7.1}$$

$\hat{G}(x',x,\omega)$ satisfies the PDE (see Eq. (6.7) in Appendix for Step 6)

$$\omega^2\hat{G}(x',x,\omega)-2\bar{u}\,\mathrm{i}\omega\frac{\partial\hat{G}}{\partial x'}+\left(c^2-\bar{u}^2\right)\frac{\partial^2\hat{G}}{\partial x'^2}=-\delta(x'-x)\,. \tag{7.2}$$

We propose the following "trial relationship",

$$G(x',x,t',t)=\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\hat{G}(x',x,\omega)\,\mathrm{e}^{-\mathrm{i}\omega(t-t')}\,\mathrm{d}\omega\,. \tag{7.3}$$

Then

$$\frac{\partial G}{\partial t'}=\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\mathrm{i}\omega\hat{G}(x',x,\omega)\,\mathrm{e}^{-\mathrm{i}\omega(t-t')}\,\mathrm{d}\omega\,, \tag{7.4a}$$

$$\frac{\partial^2 G}{\partial t'^2}=\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}(-\omega^2)\hat{G}(x',x,\omega)\,\mathrm{e}^{-\mathrm{i}\omega(t-t')}\,\mathrm{d}\omega\,, \tag{7.4b}$$

$$\frac{\partial^2 G}{\partial t'\partial x'}=\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\mathrm{i}\omega\frac{\partial\hat{G}}{\partial x'}\mathrm{e}^{-\mathrm{i}\omega(t-t')}\,\mathrm{d}\omega\,, \tag{7.4c}$$

$$\frac{\partial^2 G}{\partial x'^2}=\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\frac{\partial^2\hat{G}}{\partial x'^2}\mathrm{e}^{-\mathrm{i}\omega(t-t')}\,\mathrm{d}\omega\,. \tag{7.4d}$$

We multiply (7.2) by $\frac{1}{2\pi}\mathrm{e}^{-\mathrm{i}\omega(t-t')}$ and then integrate $\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\ldots\ \mathrm{e}^{-\mathrm{i}\omega(t-t')}\,\mathrm{d}\omega$.

This leads to

$$\underbrace{\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\omega^2\hat{G}(x',x,\omega)\mathrm{e}^{-\mathrm{i}\omega(t-t')}\,\mathrm{d}\omega}_{=-\frac{\partial^2 G}{\partial t'^2}}-2\bar{u}\underbrace{\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\mathrm{i}\omega\frac{\partial\hat{G}}{\partial x'}\mathrm{e}^{-\mathrm{i}\omega(t-t')}\,\mathrm{d}\omega}_{=\frac{\partial^2 G}{\partial t'\partial x'}}+$$

$$+\left(c^2-\bar{u}^2\right)\underbrace{\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\frac{\partial^2\hat{G}}{\partial x'^2}\mathrm{e}^{-\mathrm{i}\omega(t-t')}\,\mathrm{d}\omega}_{=\frac{\partial^2 G}{\partial x'^2}}=-\delta(x'-x)\underbrace{\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\omega(t-t')}\,\mathrm{d}\omega}_{=\delta(t'-t)}. \tag{7.5}$$

We rewrite (7.5) as indicated by the blue terms and obtain

$$\frac{\partial^2 G}{\partial t'^2} + 2\bar{u}\frac{\partial^2 G}{\partial t'\partial x'} - \left(c^2 - \bar{u}^2\right)\frac{\partial^2 G}{\partial x'^2} = \delta(x - x')\delta(t - t')\,. \tag{7.6}$$

This agrees with (7.1), and therefore our test relationship in (7.3) has been validated.

We can conclude that $G(x',x,t',t)$ and $\hat{G}(x',x,\omega)$ are a Fourier transform pair.

Appendix for Step 8: Show that the boundary term BT2 in Step 3 is zero

$$\mathrm{BT2} = \int_{t'=0}^{T_?} \left[\bar{u}\left(G\frac{\partial\phi}{\partial t'} - \phi\frac{\partial G}{\partial t'}\right) - (c^2 - \bar{u}^2)\left(G\frac{\partial\phi}{\partial x'} - \phi\frac{\partial G}{\partial x'}\right) \right]_{x'=0}^{L} \mathrm{d}t' \tag{8.1}$$

For times t' outside the integration range, we can assume that

$$\phi(x',t') = 0 \text{ for } t' < 0 \tag{8.2a}$$

$$G(x',x,t',t) = 0 \text{ for } t' > T_? \tag{8.2b}$$

This allows us to extend the integration limits from $\int_{t'=0}^{T_?}$ to $\int_{t'=-\infty}^{\infty}$.

We now focus on the integration with respect to t'.

Since $\phi(x',t')$ and $G(x',x,t',t)$ are given by

$$\phi(x',t') = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \hat{\phi}(x',\omega)\mathrm{e}^{-\mathrm{i}\omega t'}\mathrm{d}\omega \tag{8.3a}$$

$$G(x',x,t',t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \hat{G}(x',x,\tilde{\omega})\mathrm{e}^{-\mathrm{i}\tilde{\omega}(t-t')}\,\mathrm{d}\tilde{\omega} \tag{8.3b}$$

each of the product terms

$$G\frac{\partial\phi}{\partial t'} \;,\; \phi\frac{\partial G}{\partial t'} \;,\; G\frac{\partial\phi}{\partial x'} \;,\; \phi\frac{\partial G}{\partial x'}$$

in (8.1) is a double integral of the form $\int_{\omega=-\infty}^{\infty}\int_{\tilde{\omega}=-\infty}^{\infty} \ldots\, \mathrm{d}\tilde{\omega}\mathrm{d}\omega$. Each integrand of these double integrals has the same time-dependence, $\mathrm{e}^{-\mathrm{i}\omega t'}\,\mathrm{e}^{-\mathrm{i}\tilde{\omega}(t-t')} = \mathrm{e}^{-\mathrm{i}\tilde{\omega}t}\,\mathrm{e}^{-\mathrm{i}(\omega-\tilde{\omega})t'}$, which can be integrated with respect to t'.

$$\int_{t'=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\tilde{\omega}t}\,\mathrm{e}^{-\mathrm{i}(\omega-\tilde{\omega})t'}\mathrm{d}t' = \mathrm{e}^{-\mathrm{i}\tilde{\omega}t}\int_{t'=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(\omega-\tilde{\omega})t'}\mathrm{d}t' = \mathrm{e}^{-\mathrm{i}\tilde{\omega}t}\; 2\pi\delta(\omega-\tilde{\omega}) \tag{8.4}$$

As a consequence of the term $\delta(\omega-\tilde{\omega})$ in the double integral over ω and $\tilde{\omega}$ reduces to a single integral over ω. This allows us to consider a *single* frequency component of ϕ and G,

$$\phi_\omega(x',t') = \hat{\phi}(x',\omega)\mathrm{e}^{-\mathrm{i}\omega t'} \tag{8.5}$$

$$G_\omega(x',x,t',t) = \hat{G}(x',x,\tilde{\omega})\mathrm{e}^{-\mathrm{i}\tilde{\omega}(t-t')} \tag{8.6}$$

when calculating the product terms.

We first consider the boundary at $x' = 0$. Near there we have (see Eq. (5) for ϕ_ω and Eq. (17) for G_ω)

$$\phi_\omega(x',t') = A_-(R_0 e^{ik_+x'} + e^{-ik_-x'})\, e^{-i\omega t'} \tag{8.7a}$$

$$\frac{\partial\phi_\omega}{\partial t'} = (-i\omega)A_-(R_0 e^{ik_+x'} + e^{-ik_-x'})\, e^{-i\omega t'} \tag{8.7b}$$

$$\frac{\partial\phi_\omega}{\partial x'} = A_-(ik_+R_0 e^{ik_+x'} - ik_- e^{-ik_-x'})\, e^{-i\omega t'} \tag{8.7c}$$

$$G_\omega(x',x,t',t) = \tilde{A}_-(x,\omega)(R_0 e^{ik_-x'} + e^{-ik_+x'})\, e^{-i\omega(t-t')} \tag{8.8a}$$

$$\frac{\partial G_\omega}{\partial t'} = (i\omega)\tilde{A}_-(x,\omega)(R_0 e^{ik_-x'} + e^{-ik_+x'})\, e^{-i\omega(t-t')} \tag{8.8b}$$

$$\frac{\partial G_\omega}{\partial x'} = \tilde{A}_-(x,\omega)(ik_-R_0 e^{ik_-x'} - ik_+ e^{-ik_+x'})\, e^{-i\omega(t-t')} \tag{8.8c}$$

We evaluate (8.7) and (8.8) at $x' = 0$ to get

$$\phi_\omega(x',t')\Big|_{x'=0} = A_-(R_0+1)\, e^{-i\omega t'} \tag{8.9a}$$

$$\left.\frac{\partial\phi_\omega}{\partial t'}\right|_{x'=0} = -i\omega A_-(R_0+1)\, e^{-i\omega t'} \tag{8.9b}$$

$$\left.\frac{\partial\phi_\omega}{\partial x'}\right|_{x'=0} = A_-(ik_+R_0 - ik_-)\, e^{-i\omega t'} \tag{8.9c}$$

$$G_\omega(x',x,t',t)\Big|_{x'=0} = \tilde{A}_-(x,\omega)(R_0+1)\, e^{-i\omega(t-t')} \tag{8.10a}$$

$$\left.\frac{\partial G_\omega}{\partial t'}\right|_{x'=0} = (i\omega)\tilde{A}_-(x,\omega)(R_0+1)\, e^{-i\omega(t-t')} \tag{8.10b}$$

$$\left.\frac{\partial G_\omega}{\partial x'}\right|_{x'=0} = \tilde{A}_-(x,\omega)(ik_-R_0 - ik_+)\, e^{-i\omega(t-t')} \tag{8.10c}$$

Then the integrand in (8.1) becomes at $x' = 0$

$$\left[\bar{u}(G_\omega \frac{\partial \phi_\omega}{\partial t'} - \phi_\omega \frac{\partial G_\omega}{\partial t'}) - (c^2 - \bar{u}^2)(G_\omega \frac{\partial \phi_\omega}{\partial x'} - \phi_\omega \frac{\partial G_\omega}{\partial x'})\right]_{x'=0} =$$

$$= \bar{u}\left[\tilde{A}_-(R_0+1)(-\mathrm{i}\omega)\, A_-(R_0+1) - A_-(R_0+1)(\mathrm{i}\omega)\, \tilde{A}_-(R_0+1)\right] \mathrm{e}^{-\mathrm{i}\omega t'} \mathrm{e}^{-\mathrm{i}\omega(t-t')} -$$

$$-(c^2 - \bar{u}^2)\left[\tilde{A}_-(R_0+1)\, A_-(\mathrm{i}k_+ R_0 - \mathrm{i}k_-) - A_-(R_0+1)\tilde{A}_-(\mathrm{i}k_- R_0 - \mathrm{i}k_+)\right] \mathrm{e}^{-\mathrm{i}\omega t'} \mathrm{e}^{-\mathrm{i}\omega(t-t')} =$$

$$= \tilde{A}_- A_-(R_0+1)\mathrm{e}^{-\mathrm{i}\omega t}\left[\bar{u}(-2\mathrm{i}\omega)(R_0+1) - (c^2-\bar{u}^2)\underbrace{(\mathrm{i}k_+ R_0 - \mathrm{i}k_- - \mathrm{i}k_- R_0 + \mathrm{i}k_+)}_{= R_0(\mathrm{i}k_+ - \mathrm{i}k_-) + \mathrm{i}k_+ - \mathrm{i}k_- = (R_0+1)(\mathrm{i}k_+ - \mathrm{i}k_-)}\right] =$$

$$= \tilde{A}_- A_-(R_0+1)^2 \mathrm{e}^{-\mathrm{i}\omega t}\left[-2\mathrm{i}\omega\bar{u} - (c^2-\bar{u}^2)\underbrace{(\mathrm{i}k_+ - \mathrm{i}k_-)}_{= \mathrm{i}\omega \frac{-2\bar{u}}{c^2 - \bar{u}^2}}\right] = 0 \qquad (8.11)$$

The same calculation can be done for the boundary at $x' = L$, and the result is also zero.

To summarize, we have shown that

$$\mathrm{BT2} = 0 \qquad (8.12)$$

in Eq. (6).

Appendix for Step 9: Find the relationship between *g* and *G*

$g(x,x',t,t')$ satisfies the PDE (see Eq. (10))

$$\frac{\partial^2 g}{\partial t^2}+2\bar{u}\frac{\partial^2 g}{\partial t\partial x}-\left(c^2-\bar{u}^2\right)\frac{\partial^2 g}{\partial x^2}=\delta(x-x')\delta(t-t')\ , \tag{9.1}$$

and the causality condition (see Eq. (11a))

$$g(x,x',t-t')=0 \quad \text{for} \quad t<t'\ . \tag{9.2}$$

We multiply (9.1) by $G(x,x^*,t,t^*)$ and integrate $\int\limits_{t=-\infty}^{\infty}\int\limits_{x=0}^{L}\ldots G(x,x^*,t,t^*)\ldots \mathrm{d}t\,\mathrm{d}x$. This

gives

$$\underbrace{\int\limits_{t=-\infty}^{\infty}\int\limits_{x=0}^{L}\frac{\partial^2 g}{\partial t^2}G\,\mathrm{d}t\,\mathrm{d}x}_{=I_1}+\underbrace{2\bar{u}\int\limits_{t=-\infty}^{\infty}\int\limits_{x=0}^{L}\frac{\partial^2 g}{\partial t\partial x}G\,\mathrm{d}t\,\mathrm{d}x}_{=I_2}-\underbrace{\left(c^2-\bar{u}^2\right)\int\limits_{t=-\infty}^{\infty}\int\limits_{x=0}^{L}\frac{\partial^2 g}{\partial x^2}G\,\mathrm{d}x\,\mathrm{d}t}_{=I_3}=$$

$$=\underbrace{\int\limits_{t=-\infty}^{\infty}\int\limits_{x=0}^{L}\delta(x-x')\delta(t-t')G(x,x^*,t,t^*)\mathrm{d}x\,\mathrm{d}t}_{=G(x',x^*,t',t^*)} \tag{9.3}$$

The three integrals I_1, I_2, I_3, can be rewritten with integration by parts. The result for I_1 is

$$I_1=\int\limits_{x=0}^{L}\left[\frac{\partial g}{\partial t}G-\frac{\partial G}{\partial t}g\right]_{t=-\infty}^{\infty}\mathrm{d}x+\int\limits_{x=0}^{L}\int\limits_{t=-\infty}^{\infty}g\frac{\partial^2 G}{\partial t^2}G\,\mathrm{d}t\,\mathrm{d}x \tag{9.4}$$

I_2 can be manipulated in two different ways: integrate first with respect to t and then x, or vice versa. The first way gives

$$I_2=\int\limits_{x=0}^{L}\left[\frac{\partial g}{\partial x}G\right]_{t=-\infty}^{\infty}\mathrm{d}x-\int\limits_{t=-\infty}^{\infty}\left[g\frac{\partial G}{\partial t}\right]_{x=0}^{L}\mathrm{d}t+\int\limits_{t=-\infty}^{\infty}\int\limits_{x=0}^{L}g\frac{\partial^2 G}{\partial t\partial x}\mathrm{d}x\,\mathrm{d}t\ . \tag{9.5a}$$

The second way gives

$$\tilde{I}_2=\int\limits_{t=-\infty}^{\infty}\left[\frac{\partial g}{\partial t}G\right]_{x=0}^{L}\mathrm{d}t-\int\limits_{x=0}^{L}\left[g\frac{\partial G}{\partial x}\right]_{t=-\infty}^{\infty}\mathrm{d}x+\int\limits_{x=0}^{L}\int\limits_{t=-\infty}^{\infty}g\frac{\partial^2 G}{\partial t\partial x}\mathrm{d}t\,\mathrm{d}x\ . \tag{9.5b}$$

The result for I_3 is

$$I_3=\int\limits_{t=-\infty}^{\infty}\left[\frac{\partial g}{\partial x}G-g\frac{\partial G}{\partial x}\right]_{x=0}^{L}\mathrm{d}t+\int\limits_{t=-\infty}^{\infty}\int\limits_{x=0}^{L}g\frac{\partial^2 G}{\partial x^2}\mathrm{d}x\,\mathrm{d}t\ . \tag{9.6}$$

We now substitute (9.4) – (9.6) into (9.3).

$$G(x',x^*,t',t^*) = I_1 + 2\bar{u}I_2 - (c^2 - \bar{u}^2)I_3 =$$

$$= I_1 + \bar{u}I_2 + \bar{u}\tilde{I}_2 - (c^2 - \bar{u}^2)I_3 =$$

$$= \int_{x=0}^{L} \int_{t=-\infty}^{\infty} g(x,x',t-t') \underbrace{\left[\frac{\partial^2 G}{\partial t^2} + 2\bar{u}\frac{\partial^2 G}{\partial t \partial x} - (c^2 - \bar{u}^2)\frac{\partial^2 G}{\partial x^2}\right]}_{=\delta(t-t^*)\delta(x-x^*)} \mathrm{d}t\,\mathrm{d}x +$$

$$+ \int_{x=0}^{L} \left\{ \left[\frac{\partial g}{\partial t}G - \frac{\partial G}{\partial t}g\right]_{t=-\infty}^{\infty} + \bar{u}\left[\frac{\partial g}{\partial x}G\right]_{t=-\infty}^{\infty} - \bar{u}\left[g\frac{\partial G}{\partial x}\right]_{t=-\infty}^{\infty} \right\} \mathrm{d}x +$$

$$+ \int_{t=-\infty}^{\infty} \left\{ -\bar{u}\left[g\frac{\partial G}{\partial t}\right]_{x=0}^{L} + \bar{u}\left[\frac{\partial g}{\partial t}G\right]_{x=0}^{L} - (c^2 - \bar{u}^2)\left[\frac{\partial g}{\partial x}G - g\frac{\partial G}{\partial x}\right]_{x=0}^{L} \right\} \mathrm{d}t =$$

$$= g(x^*,x',t^*,t') +$$

$$+ \int_{x=0}^{L} \underbrace{\left[G\left(\frac{\partial g}{\partial t} + \bar{u}\frac{\partial g}{\partial x}\right) - g\left(\frac{\partial G}{\partial t} + \bar{u}\frac{\partial G}{\partial x}\right)\right]_{t=-\infty}^{\infty}}_{=\mathrm{BT4}} \mathrm{d}x +$$

$$\underbrace{+ \int_{t=-\infty}^{\infty} \left[\bar{u}\left(\frac{\partial g}{\partial t}G - g\frac{\partial G}{\partial t}\right) - (c^2 - \bar{u}^2)\left(\frac{\partial g}{\partial x}G - g\frac{\partial G}{\partial x}\right)\right]_{x=0}^{L} \mathrm{d}t}_{=\mathrm{BT5}} \qquad (9.7)$$

We will get a meaningful result if we can show that the boundary terms BT4 and BT5 are zero. In order to determine BT4, we use the causality condition and terminal condition. From the causality condition (9.2), we can conclude that

$$g(x,x',t-t') = 0 \quad \text{for} \quad t = -\infty, \qquad (9.8)$$

because t' is finite. From the terminal condition, which we extended in (8.2), we can conclude that

$$G(x,x^*,t,t^*) = 0 \quad \text{for} \quad t = \infty, \qquad (9.9)$$

because T_2 is finite. As a consequence of (9.8) and (9.9),

$$\mathrm{BT4} = 0. \qquad (9.10)$$

The term BT5 in (9.7) is analogous to the term BT2 in (8.1), with g in place of ϕ. In the frequency domain, g and ϕ have the same wave numbers (k_+ for forward travelling waves; $k\_$ for backward travelling waves) and the same boundary conditions at the tube ends $x=0,L$. Hence the method applied in Step 8 can be applied again in this case, leading to

$$\mathrm{BT5}=0\,. \tag{9.11}$$

We can now conclude from (9.7), (9.10) and (9.11) that

$$G(x',x,t',t)=g(x,x',t,t')\,. \tag{9.12}$$

This result expresses *the reciprocity* between the direct and adjoint Green's function [Morse and Feshbach 1953, section 7.5]. We note that neither G , nor g , are symmetric,

$$g(x,x',t-t')\neq g(x',x,t'-t)\,, \tag{9.13}$$

$$G(x',x,t'-t)\neq G(x,x',t-t')\,. \tag{9.14}$$

Appendix for Step 11: Fix the terminal time

The integral equation (9) reduces with $\mathrm{BT2} = 0$ (see Appendix for Step 8) to

$$\phi(x,t) = -\frac{\gamma - 1}{\overline{\rho}} \int_{t'=0}^{T_?} G(x_q, x, t', t) q(t') \, \mathrm{d}t' + \left[\varphi'_0 G(x', x, t', t) - \varphi_0 \left(\frac{\partial G}{\partial t'} + \overline{u} \frac{\partial G}{\partial x'} \right) \right]_{\substack{x'=x_q \\ t'=0}} \quad (11.1)$$

Given that $G = 0$ for all times $t > t'$, the integrand is zero in the range $t' = t, \ldots T_?$. This suggests that the upper integration boundary should be changed from $T_?$ to t.

The final version of the integral equation is

$$\phi(x,t) = -\frac{\gamma - 1}{\overline{\rho}} \int_{t'=0}^{t} G(x_q, x, t', t) q(t') \, \mathrm{d}t' + \left[\varphi'_0 G(x', x, t', t) - \varphi_0 \left(\frac{\partial G}{\partial t'} + \overline{u} \frac{\partial G}{\partial x'} \right) \right]_{\substack{x'=x_q \\ t'=0}} \quad (11.2)$$

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